Semi-classical limits of the first eigenfunction and concentration on the recurrent sets of a dynamical system

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Abstract

We study the semi-classical limits of the first eigenfunction of a positive second order operator on a compact Riemannian manifold, when the diffusion constant $\epsilon$ goes to zero. If the drift of the diffusion is given by a Morse-Smale vector field $b$, the limits of the eigenfunctions concentrate on the recurrent set of $b$. A blow-up analysis enables us to find the main properties of the limit measures on a recurrent set.

We consider generalized Morse-Smale vector fields, the recurrent set of which is composed of hyperbolic critical points, limit cycles and two dimensional torii. Under some compatibility conditions between the flow of $b$ and the Riemannian metric $g$ along each of these components, we prove that the support of a limit lies on those recurrent components of maximal dimension, where the topological pressure is achieved. Also, the restriction of the limit measure to either a cycle or a torus is absolutely continuous with respect to the unique invariant probability measure of the restriction of $b$ to the cycle or the torus. When the torii are not charged, the restriction of the limit measure is absolutely continuous with respect to the arclength on the cycle and we have determined the corresponding density. Finally, the support of the limit measures and the support of the measures selected by the variational formulation of the topological pressure (TP) are identical.

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1 Introduction

A random particle can escape in finite time from the basin of attraction of a dynamical system [17, 38]. Indeed a large fluctuation will ultimately push the stochastic particle outside of the basin even if the drift points inside. On the other hand, on a compact manifold, the analysis of the motion of a stochastic particle is quite different. In particular the particle can wander inside the manifold from one basin of attraction to another. We are interested here in finding the behavior of the random particle for large time and small noise.

The answer to this problem depends on two main data, at least. First the geometrical space which is a Riemannian manifold and second a dynamical system. It is legitimate to think that the recurrent set of the field plays a crucial role since they are visited repeatedly by the deterministic particle. One can ask, for example, if certain recurrent sets are visited more often than others. To address this question, one would try to obtain an explicit solution of the Fokker-Planck Equation (FPE) and derive large time and small noise asymptotics. We shall study the FPE and in particular the ground state (first eigenfunction) when the noise is small.

1.0.1 Formulation of the first eigenfunction problem

On compact Riemannian manifolds, the behavior of the first eigenvalue of singular elliptic perturbations of first order operators has been the topic of many studies (among others [9, 17]). As the viscosity parameter goes to zero, the first eigenvalue tends to a quantity called the topological pressure. We shall discussed briefly the concept developed in [31] and link it with the limits of the first eigenfunctions.

In a previous study [24], when the first order term is a Morse-Smale field and a killing potential \( c \) is added to the dynamic, some properties of the limit measures of the first eigenfunction were obtained. It was shown that the limit of a normalized eigenfunction is concentrated on a subset of the recurrent set (see [24]). There, we have left open the characterizations of the limit measures. The purpose of the present work is to complete and extend this previous study [24] and to describe in some cases the properties of these measures. In order to make our exposition self contained, we shall recall briefly the links between the dynamics of a random particle and its survival probability distribution.

If \( X_\epsilon(t) \) denotes the position at time \( t \) of a random particle on a compact Riemannian manifold \((V,g)\) its motion is described by the following stochastic equation:

\[
dX_\epsilon(t) = b(X_\epsilon(t))dt + \sigma(X_\epsilon(t))dw
\]

where \( w \) is the N-dimensional classical Brownian process and \( \sigma \) is a vector bundle homomorphism from the trivial vector bundle \( V \times \mathbb{R}^N \) into the tangent bundle \( T_V \) such that \( \sigma \sigma^* = \tilde{g}^{-1} \) where \( \sigma^* : T^* V \rightarrow V \times \mathbb{R}^N \) is the homomor-
phism dual to $\sigma$ and $\tilde{g} : TV \rightarrow T^*V$ is the canonical fiber bundle isomorphism induced by the metric $g$. A potential $c$ representing a killing term is added. Physical interpretations of $c$ and computation of the survival probability can be found in [38, 25]. The survival probability distribution of a particle $X_\epsilon(t)$ located in the region $y + dy$ at time $t$, conditioned by the initial condition $X_\epsilon(0) = x$ (see [38]) is given by $p_\epsilon(t, x, y) = \Pr(X_\epsilon(t) \in y + dy | X_\epsilon(0) = x)$ and satisfies the backward Fokker-Planck equation (FPE):

$$\frac{\partial p_\epsilon(t, x, y)}{\partial t} = \varepsilon \Delta_g p_\epsilon(t, x, y) + \langle b(x), \nabla p_\epsilon(t, x, y) \rangle + c(x)p_\epsilon(t, x, y)$$

$$p_\epsilon(0,..,y) = \delta_y.$$

We consider the Fokker Planck operator defined by

$$L_\epsilon = \varepsilon \Delta_g + \theta(b) + c,$$

(1)

where $\Delta_g$ is the Laplace-Beltrami operator and $\theta(b)$ is the Lie derivative in the direction $b$. The function $c$ is chosen such that $L_\epsilon$ is positive. $L_\epsilon$ cannot be conjugated to a self-adjoint operator by scalar multiplication. Since the operator is positive and compact, by the Krein-Rutman theorem the first eigenvalue $\lambda_\epsilon$ is real positive and associated to a unique positive normalized eigenfunction $u_\epsilon$ (see for example [36]).

The probability density function $p_\epsilon$ can be expanded using the eigenfunction of $L_\epsilon$

$$p_\epsilon(t, x, y) = e^{-\lambda_\epsilon t}u_\epsilon(x)u_\epsilon^*(y) + R_\epsilon(t, x, y),$$

where the second term $R_\epsilon(t, x, y)$ decreases exponentially faster than the first as $t$ goes to infinity, and $u_\epsilon^*$ is the first positive formalized eigenfunction associated to the adjoint operator $L_\epsilon^*$. The first normalized positive eigenfunction $u_\epsilon > 0$, is a solution of

$$\varepsilon \Delta_g u_\epsilon + \langle b, \nabla u_\epsilon \rangle + c u_\epsilon = \lambda_\epsilon u_\epsilon$$

$$\int_{V_n} u_\epsilon^2 dV_g = 1.$$

Equation (2) has a long story in the literature, starting from the work in the 70’s in $\mathbb{R}^n$ to more recent endeavors, to find a quantum analog of the weak KAM theory (see for example [15]). In that case, when the field $b$ is Hamiltonian, the measure $u_\epsilon u_\epsilon^* dV_g$ as $\epsilon$ goes to zero concentrates on a specific set.

1.0.2 Limit of the first eigenvalue and the topological pressure

Under hyperbolicity assumptions on the field $b$, the limit of the sequence $\lambda_\epsilon$ has been determined in [10, 31]. It was based on the fact that $\lambda_\epsilon$ is given by the following variational formula (see [10]),

$$\lambda_\epsilon = \sup_{\mu \in P(V)} \left( \int_V c d\mu + \inf_{u > 0} \left[ -\int_V \frac{L_\epsilon(u)}{u} d\mu \right] \right),$$
where \( \mu \) belongs to \( P(V) \), the space of probability measure on \( V \). It is proved in [31] that when the recurrent set \( K \) of the field \( b \) is a finite union of isolated components \( K_1, K_2, ..., K_N \) which are hyperbolic invariant sets, the limit of the first eigenvalue \( \lambda_\epsilon \) as \( \epsilon \) goes to zero is equal to topological pressure, denoted by \( TP \) and defined as follows:

\[
TP(K_j) = \sup_{\mu} \left\{ h_\mu + \int_V \left( c - \frac{d \det D\phi_t^\mu}{dt} \bigg|_{t=0} \right) d\mu \mid \mu \in P(V), \text{support } \mu \subset K_j, \mu \text{ } \phi_t - \text{invariant} \right\}
\]

For any union \( U \) of \( K_j \)'s,

\[
TP(U) = \sup_{K_j \in U} TP(K_j), \text{ where } h_\mu \text{ is the metric entropy (see [37]), } \phi_t \text{ is the flow of the vector field } b \text{ and } D\phi_t^\mu \text{ is the tangent mapping of } \phi \text{ restricted to the unstable bundle } T^u_{K_j}V \text{ of } K_j \text{ and } \det D\phi_t^\mu \text{ is the determinant of } D\phi_t^\mu \text{ with respect to the metric } g. \text{ A } \phi_t - \text{invariant measure } \mu_j \in P(V) \text{ with support in } K_j \text{ such that }
\]

\[
TP(K_j) = h_{\mu_j} + \int_V \left( c - \frac{d \det D\phi_t^\mu}{dt} \bigg|_{t=0} \right) d\mu_j
\]

is called an equilibrium state associated to \( K_j \) (see [31],[10] for existence). Up to a reordering of \( K_j \) we can assume that \( TP(K_j) = TP(\cup_i K_i) \) if \( 1 \leq j \leq l \), and \( TP(K_j) < TP(\cup_i K_i) \) if \( j > l \).

A measure \( \mu = \sum_{j=1}^l p_j \mu_{K_j} \) where \( \mu_{K_j} \) is an equilibrium state associated with the set \( K_j \), \( p_j \geq 0, \sum_{j=1}^l p_j = 1 \), is called an equilibrium measure (see Theorem 3.4 in [32] p.20). Unfortunately, neither these results nor the methods used in [31], Donsker-Varadhan [10]-[11]-[12] and many others do give us any information about the first eigenfunction \( u_\epsilon \) as \( \epsilon \) goes to zero.

1.0.3 Limit of the first eigenfunction

Let us introduce the following notations,

\[
b = \Omega + \nabla L, \quad v_\epsilon = e^{-\frac{\epsilon}{2}} u_\epsilon, \quad (3)
\]

\[
c_\epsilon = \epsilon \left( c + \frac{\Delta g L}{2} \right) + \Psi L,
\]

\[
\Psi L = \frac{1}{4} \left( ||\nabla L||^2 + 2(\nabla L, \Omega) \right),
\]

where the function \( L \) is a special Lyapunov function (see appendix I of [24], for a construction of \( L \) associated with a Morse-Smale vector field on a compact Riemannian manifold). The introduction of the function \( L \) is natural because
in the neighborhood of any recurrent sets of the field $b$, it coincides to the third order with the solution of the associated Hamilton-Jacobi equation ($\Psi_{L} = 0$). This solution has already been introduced in the classical text books by Courant-Hilbert to find approximations to the solutions of the Wave equation, and later on by Schuss [38], Kamin [28, 29, 30], Friedman [19], and many other, in the context of the diffusion equation. The exponential of the Lyapunov function plays the role of a convergence factor, which is a well established technic in analysis especially in the study of divergent series and integrals.

Equation (2) is transformed into

$$L_{\epsilon}(v_{\epsilon}) = \epsilon^{2}\Delta_{g}v_{\epsilon} + \epsilon(\Omega, \nabla v_{\epsilon}) + c_{\epsilon}v_{\epsilon} = \epsilon\lambda_{\epsilon}v_{\epsilon}, \tag{4}$$

and we impose the normalization condition

$$\int_{V_{n}} v_{\epsilon}^{2}dV_{g} = 1.$$

In equation (4), $\epsilon\lambda_{\epsilon}$ is the first eigenvalue of the operator

$$L'_{\epsilon} = \epsilon^{2}\Delta_{g} + \epsilon\theta(\Omega) + c_{\epsilon}$$

and $v_{\epsilon}$ is the associated positive [36] eigenfunction. It has been proved in [24] that the weak limits of the normalized measures $v_{\epsilon}^{2} dV_{g}$ are supported by the limit sets of the field $b$. In order to obtain a precise description of these limits additional assumptions will be made on the behavior of the vector field $b$ near the hyperbolic recurrent sets. Usually the first eigenfunction $u_{\epsilon}$ will not have any limits as $\epsilon$ goes to zero in any of the classical "strong" topologies. On the other hand we will characterize the limits of the measure $v_{\epsilon}^{2}dV_{g}$ if we assume that $b$ is a generalized Morse-smale field, the only recurrent components of which are hyperbolic points, cycles, two dimensional torii and the hyperbolic structure satisfies some compatibility conditions with the metric. The main result of this work can be summarized as follow:

"On a Riemannian manifold, for any choice of a special Lyapunov function $L$, vanishing at order 2 on the recurrent sets of the field, the limits as $\epsilon$ tends to 0, of the normalized measures $e^{-\frac{\epsilon}{2}u_{\epsilon}^{2}dV_{g}}$, are concentrated on the components of the recurrent sets which are of maximal dimension and where the topological pressure is achieved."

1.1 Notations and Assumptions

We shall make the following assumptions on the field $b$:

1) The recurrent set is a finite union of stationary points, limit cycles and two dimensional torii.
II) The stationary points are hyperbolic and for each such $P$ the stable and unstable manifolds belonging to $P$ are orthogonal at $P$ with respect to the metric $g$.

III) Any limit cycle $S$ has a tubular neighborhood $T^S$ equipped with a covering map $\Phi : \mathbb{R}^{m-1} \times \mathbb{R} \rightarrow T^S$ having the following properties:
(a) for all $(x', \theta) \in \mathbb{R}^{m-1} \times \mathbb{R}$,

$$\Phi^{-1} \circ \Phi(x', \theta) = \{(x', \theta + nT_S)| n \in \mathbb{Z}\}$$

where $T_S$ is the minimal period of the cycle $S$.
(b) at any point $(0, \theta) \in \mathbb{R}^{m-1} \times \mathbb{R}$,

$$(\Phi)^* g_{(0, \theta)} = \sum_{n=1}^{m-1} dx_n^2 + g^{mm}(\theta) d\theta^2,$$

where $\theta = x^m$.
(c) at any point $(0, \theta) \in \mathbb{R}^{m-1} \times \mathbb{R}$,

$$(\Phi)_* b = \frac{\partial}{\partial \theta} + \sum_{i,j=1}^{m-1} B_{ij} x_j \frac{\partial}{\partial x_i},$$

up to term of order two in $x' = (x_1, .., x_{m-1})$, canonical coordinates on $\mathbb{R}^{m-1}$.
(d) the $(m - 1) \times (m - 1)$ matrix $B = \{B_{ij}|1 \leq i,j \leq m-1\}$ is hyperbolic and its stable and unstable spaces are orthogonal with respect to the Euclidean metric $\sum_{n=1}^{m-2} x_n^2$ on $\mathbb{R}^{m-1}$. Denote by $B^s$ (resp. $B^u$) the restriction of $B$ to the stable (resp unstable) space.

IV) Any 2-dimensional torus $R$ has a tubular neighborhood $T^R$ equipped with a diffeomorphism $\Phi : \mathbb{R}^{m-2} \times \mathbb{R}^2 \rightarrow T^R$ having the following properties:
(a) at any point $(0, \theta) \in \mathbb{R}^{m-2} \times \mathbb{R}^2$, $\theta = (\theta_1, \theta_2)$ $\theta_1, \theta_2$ cyclic coordinates, with period one.

$$(\Phi^{-1})^* g_{(0, \theta)} = \sum_{n=1}^{m-2} dx_n^2 + a(\theta)d\theta_1^2 + 2b(\theta)d\theta_1d\theta_2 + c(\theta)d\theta_2^2.$$

(b) at any point $(0, \theta) \in \mathbb{R}^{m-2} \times \mathbb{R}^2$,

$$(\Phi)_* (b)(0, \theta) = k_1 \frac{\partial}{\partial \theta_1} + k_2 \frac{\partial}{\partial \theta_2} + \sum_{i,j=1}^{m-2} B_{ij} x_j \frac{\partial}{\partial x_i},$$

where:
(i) the $(m - 2) \times (m - 2)$ matrix $B = \{B_{ij}|1 \leq i,j \leq m-2\}$ is hyperbolic and its stable and unstable spaces are orthogonal with respect to the Euclidean metric $\sum_{n=1}^{m-2} x_n^2$ on $\mathbb{R}^{m-2}$.
(ii) \( k_1, k_2 \in \mathbb{R} \) and \( \frac{k_1}{k_2} \in \mathbb{R} - \mathbb{Q} \). A torus with such a flow will be called an irrational torus.

(iii) there exist constants \( C > 0, \alpha > 0 \) such that for all \( m_1, m_2 \in \mathbb{N} \),
\[
|m_1 k_1 + m_2 k_2| > C (m_1^2 + m_2^2)^\alpha
\]
This is usually called the small divisor condition. We call assumption (i) in all the previous cases the **Orthogonality Assumption**.

**Definition and construction of a special Lyapunov function**

Given a Riemannian manifold \((V, g)\), of dimension \( m \) and a vector field \( b \) on \( V \), having as recurrent components stationary points, limit cycles and two-dimensional torii satisfying hyperbolicity conditions and compatibility conditions with \( g \), stated as in paragraph 1.1, then there exists a Lyapunov function \( L \) satisfying the following properties

1. Outside the recurrent sets, \( dL(b) < 0 \).

2. In the neighborhood of any recurrent elements \( S \) (points, cycles and torii), in the coordinate system defined in 1.1, the first nonzero term of the Taylor expansion of \( L \) is a quadratic form \( L(x) = \langle A(S)x, x \rangle_{\mathbb{R}^{m-\sigma}} + \mathcal{O}(|x|^3) \). \( \sigma \in \{0, 1, 2\} \), is the dimension of the recurrent components.

3. In the splitting of \( \mathbb{R}^{m-\sigma} = \mathbb{R}^s \oplus \mathbb{R}^u, \sigma \in \{0, 1, 2\} \), the matrix \( A(S) \), splits into \( A_s(S) \) and \( A_u(S) \). They have the form: in the system \((U, x_1, ..., x_m)\).
\[
A_s^{-1}(S) = -\int_0^{+\infty} e^{tB_s}\Pi_s e^{tB_s^*} dt,
\]
\[
A_u^{-1}(S) = \int_0^{+\infty} e^{-tB_u}\Pi_u e^{-tB_u^*} dt,
\]
where \( \Pi_s, \Pi_u \) are positive definite. For later purposes, we take \( \Pi_u >> 2Id_{m_u} \) and \( \Pi_s >> 2Id_{m_s} \). The symbol >> denote the canonical order on the set of symmetric matrices.

4. \( \Psi(L) = \frac{1}{4} (||\nabla L||_g^2 + 2 < \nabla L, \Omega >_g) = \frac{1}{4} (-||\nabla L||_g^2 + 2 < \nabla L, b >_g) \geq 0 \),
where equality occurs only on the recurrent sets. Moreover,
\[
\Psi(L) = \frac{1}{2} < B^* A(S) + A(S)B - 2A(S)^2 x, x >_{\mathbb{R}^{m-\sigma}} + \mathcal{O}(|x|^3),
\]
where the conditions on \( \Pi_s, \Pi_u \) given in 3 implies that
\[
B^* A(S) + A(S)B - 2A(S)^2 >> 0.
\]
Remark. From the conditions above, it may be surprising that all we need for our study is the knowledge of $g$ and $b$ up to the first order along the recurrent set.
1.1.1 General Notations

\[ <,>_g := \text{scalar product associated to } g \]
\[ d_g : V \times V \to \mathbb{R}_+ := \text{distance associated to } g \]
\[ \exp_x : T_xV \mapsto V := \text{exponential map of } g \text{ with pole } x \]
\[ dV_g := \text{volume element associated to } g \]
\[ L^2(V) := \text{space associated to } dV_g \]
\[ H_1([0,t]; V) := \text{the space of all } H_1 \text{ curve from } [0,t] \text{ to } V \]
\[ \Delta_g := \text{negative Laplacian associated to the metric } g \]
\[ b := \text{vector field on } V \]
\[ \theta(b) := \text{Lie derivation operator associated to } b \]
\[ \nabla := \text{gradient associated to } g \]
\[ P(V) := \text{space of all probability measures on } V \]

\( C^\infty \)-topology: = uniform convergence of all the derivatives on compact sets

\( TP := \text{Topological Pressure} \)

\[ g := \sum_{ij=1}^{m} g_{ij}dx_i dx_j \]
\[ \Delta_g := -\frac{1}{\sqrt{\det(g)}} \sum_{ij=1}^{m} \frac{\partial}{\partial x_i} \sqrt{\det(g)} g^{ij} \frac{\partial}{\partial x_j} \]
\[ g^{ij} := \text{inverse matrix of } g_{ij} \]
\[ \Delta_E^{m-1} := -\sum_{i=1}^{m-1} \frac{\partial^2}{\partial x_i^2} \]
\[ \det(g) := \det (g_{ij}) \]
\[ \Gamma^k_{ij} := \text{Christoffel symbols of } g_{ij} \]
\[ \Gamma^k_{ij} = \frac{1}{2} g^{kl} \left( \frac{\partial g_{il}}{\partial x_j} + \frac{\partial g_{jl}}{\partial x_i} - \frac{\partial g_{ij}}{\partial x_l} \right) \]
\[ R^l_{ij,k} := \frac{\partial \Gamma^l_{jk}}{\partial x_i} - \frac{\partial \Gamma^l_{ik}}{\partial x_j} + \sum_{n=1}^{m} \left[ \Gamma^l_{m,n} \Gamma^n_{jk} - \Gamma^l_{j,n} \Gamma^n_{ik} \right] \]
\[ R_{ijkl} := \sum_{n=1}^{m} g_{in} R^{'n}_{ijk} \]
\[ \text{Ric}_{kl} := \sum_{j=1}^{m} R_{jklj} \]
\[ R := \sum_{j=1}^{m} \text{Ric}_{jj} = \sum_{i,j}^{m} R_{ijji} \]
\[ \theta(b) := \sum_{i=1}^{m} b^l \frac{\partial}{\partial x^l} \]
\[ \frac{m}{\sum_{i=1}^{m} b^l \frac{\partial}{\partial x^l}} \]
Along a limit cycle $S$, parametrized by the trajectory $x_s : \mathbb{R} \to V$, we denote by $< g >_S$ the average of any regular function $g$,

$$< g >_S = \frac{1}{T_S} \int_0^{T_S} g(x_s(\theta))d\theta,$$

where $T_S$ is the minimal period of the limit cycle. Finally we denote

$$\bar{g} = g - < g >_S.$$

### 1.1.2 Expression of the Topological Pressure in some cases.

The topological pressure of the recurrent components $\omega$ of $b$ are given by the following formula

$$TP(\omega) = \pi_\omega + R(\omega).$$

where

$$\pi_\omega = \sum_i \min(0, \text{Re}\lambda_i(\omega)),$$

where $\lambda_i(\omega)$ are the eigenvalues of the matrix $B$ (see paragraph 1.1) and

- (i) if $\omega$ stationary point, $R(\omega) = c(\omega)$,
- (ii) if $\omega$ is a cycle parametrized by $\theta$ (notation 1.1),
  $$R(\omega) = \frac{1}{T_\omega} \int_0^{T_\omega} c(\theta)d\theta.$$

- (iii) if $\omega$ is a 2-dimensional irrational torus, in the coordinate system defined in the paragraphs 1.1,
  $$R(\omega) = \frac{1}{k_1 k_2} \int_{T^2} c(\theta_1, \theta_2)d\theta_1d\theta_2.$$  

where $c(\theta_1, \theta_2)$ is the restriction of $c$ to $\omega$.

The topological pressure of the field $b$ is

$$TP = \max\{\pi_\omega + R(\omega) | \omega \text{ recurrent components of } b\}.$$
1.2 Definition of Concentration Phenomena

We shall say that a limit of a measure \( \frac{v^2 dV_g}{\int_V v^2 dV_g} \) is concentrated on a set \( S \) if \( S \) has an open neighborhood \( U \) such that the support of the restriction of a limit measure to \( U \) is \( S \). The total mass of the restriction is called the concentration coefficient.

**Definition 1** For all \( \delta \) small enough fixed, \( B_P(\delta) \) is the geodesic ball of radius \( \delta \), centered at \( P \), the concentration coefficient \( c_P \) is,

\[
c_P = \lim_{n \to \infty} \frac{\int_{B_P(\delta)} v_{\epsilon_n}^2 dV_g}{\int_V v_{\epsilon_n}^2 dV_g}.
\]

There exists \( r > 0 \) such that the restriction of the limit measure to the ball \( B_P(r) \) is \( c_P \delta_P \).

This coefficient characterizes the concentration measure at point \( P \). Similarly, for a set \( S \), which can be a cycle \( \Gamma \) or torus \( T \), we have

**Definition 2** If \( \mu \) is a limit measure for a small enough \( \delta \), such that \( \text{supp} \mu \cap T^S(\delta) = S \), \( (T^S(\delta) \text{ the tubular neighborhood of } S) \) and the concentration coefficient \( c_S(\mu) \) is defined by:

\[
\text{if } v_{\epsilon_n}^2 dV_g \to \mu \text{ then } c_S(\mu) = \mu(T^S(\delta)) = \lim_{n \to \infty} \frac{\int_{T^S(\delta)} v_{\epsilon_n}^2 dV_g}{\int_V v_{\epsilon_n}^2 dV_g}.
\]

It depends on the limit \( \mu \), but not on \( \delta \) if small enough.

We will need the following additional definitions. Let us denote by \( Q_{v_{\epsilon_n}}^U \) a maximum point of \( v_{\epsilon_n} \) in \( U \).

**Definition 3** A sequence \( v_{\epsilon_n} \) where \( \epsilon_n \) tends to zero is said to charge a set \( S \) if the following limit exists and

\[
\gamma_S = \lim_{\epsilon_n \to 0} \frac{v_{\epsilon_n}(Q_{v_{\epsilon_n}}^U)}{\max_v v_{\epsilon_n}} > 0.
\]

\( \gamma_S \) is called a modulating coefficient. Note that this coefficient depends on the subsequence. Such sequence is called a charging sequence. Finally, a sequence \( v_{\epsilon_n} \) where \( \epsilon_n \) tends to zero is said to maximally charge a set \( S \) if

\[
\gamma_S = 1.
\]

We remark that the definition of \( \gamma_S \) does not depend on the open set \( U_S \), small enough.
2 Main Results

Theorem 1 On a compact Riemannian manifold \((M,g)\), let \(b\) be a Morse-Smale vector field and \(L\) be a special Lyapunov function for \(b\). Consider the normalized positive eigenfunction \(u_\epsilon > 0\) of the operator \(L_\epsilon = \epsilon \Delta + \theta(b) + c\), associated to the first eigenvalue \(\lambda_\epsilon\).

1. The recurrent set \(R\) of \(b\) is a union of a finite set of stationary points \(R_s\), a finite set of periodic orbits \(R_p\) and a finite set of two dimensional irrational torii \(R_t\). The limit set of a normalized measure

\[
\frac{u_\epsilon^2 e^{-L_\epsilon/\epsilon} dV}{\int_{V_n} u_\epsilon^2 e^{-L_\epsilon/\epsilon} dV},
\]

is contained in the set of probability measure \(\mu\) of the form

\[
\mu = \sum_{P \in R_{s\epsilon}} c_P \delta_P + \sum_{\Gamma \in R_{p\epsilon}} a_\Gamma \delta_\Gamma + \sum_{T \in R_{t\epsilon}} b_T \delta_T
\]

where \(R_{s\epsilon}\) (resp.\(R_{p\epsilon}, R_{t\epsilon}\)) is the subset of \(R_s\) (resp.\(R_p, R_t\)), where the topological pressure is attained. \(\delta_P\) is the Dirac measure at \(P\).

For \(\Gamma \in R_p\) and \(h \in C(V)\),

\[
\delta_\Gamma(h) = \int_0^{T_\Gamma} f_\Gamma(\theta) h(\Gamma(\theta)) d\theta,
\]

(6)

where \(\theta \in \mathbb{R} \rightarrow \Gamma(\theta) \in V\) is a solution of \(b\) representing \(\Gamma\) (see the notations for the precise definition of \(\theta\)). The periodic function \(f_\Gamma\) is given by

\[
f_\Gamma(\theta) = \exp\{-\int_0^\theta c(\Gamma(s)) ds + \frac{\theta}{T_\Gamma} \int_0^{T_\Gamma} c(\Gamma(s)) ds\}
\]

and \(T_\Gamma\) is the minimal period of \(\Gamma\).

For \(T \in R_t\) and \(h \in C(V)\),

\[
\delta_T(h) = \int_T h(\theta_1, \theta_2) f_T(\theta_1, \theta_2) dS_T,
\]

(7)

where \(dS_T\) is the unique probability measure on \(\mathbb{T}\) invariant under the action of the field \(b\) and \(f_T\) is the unique solution of maximum 1, of the equation

\[
k_1 \frac{\partial f}{\partial \theta_1} + k_2 \frac{\partial f}{\partial \theta_2} + cf = \mu_2 f \text{ where } \mu_2 = \int_T cdS_T.
\]
2. The coefficients $c_P, a_\Gamma, b_T$ obey the following rule: If at least one coefficient $b_T > 0$, then for all cycle $a_\Gamma = 0$ and all points $c_P = 0$. If all coefficients $b_T = 0$, and at least one coefficient $a_\Gamma > 0$ then all $c_P = 0$.

3. The limit measures of the first eigenfunction are concentrated on the recurrent set $\mathcal{R}$ of $\Omega$, where the topological pressure is attained. If $\omega \in \mathcal{R}$, and $TP(\omega) \neq TP(\mathcal{R})$, where $TP(\omega)$ is the topological pressure at $\omega$, then the limit measure associated with the normalized eigenfunction 
\[ \frac{e^{-L/\epsilon}u^2dV_g}{\int_{V_n}e^{-L/\epsilon}u^2dV_g} \] has no contribution on $\omega$.

**Remarks.**

In this last theorem, we consider only two dimensional torii such that the restriction of the vector field to the torus is diffeomorphic to an irrational flow. We do not know what should be the equivalent theorem for a field where the recurrent set contains a two dimensional surface $\Sigma$ of arbitrary genus. If the recurrent set contains a two dimensional sphere, any field contains critical points. Since there is no ergodic field on $S^2$, this suggests that the concentration of the first eigensequence cannot be absolutely continuous with respect to the surface of the sphere but should occur on some subsets $S^2$, such as the critical points. For a general Riemannian surface, it is not known how to construct an ergodic field on it and how the concentration occurs on such a surface.

When a recurrent set is a Riemannian surface $\Sigma$ which contains a limit cycle, it may be that the concentration occurs on the limit cycle rather than on the entire surface. Using a stability argument, we expect the concentration to occur on a minimal recurrent set of maximum dimension. We have not made any investigation in that fascinating direction.

It is a very interesting problem to determine the limiting weight $c_P, a_\Gamma, b_T$ of each component. Actually, even the uniqueness of the weight is still unknown. It is indeed a very difficult problem and no much results about this problem are mentioned in literature. The results presented in this paper constitute a first contribution toward the solution of that problem.

### 2.1 Description of the method

We prove the main results in several steps. First by using gauge transformation involving a special Lyapunov function $L$, the first order term in the partial differential operator given by expression (1) is transformed into one which becomes arbitrarily small as $\epsilon$ goes to zero.

The second step is the blow-up analysis of the eigenfunction. In [24] it has been proved that the concentration occurs on the recurrent sets of the field, and that the supports of the limit measures are contained in the set where the function $\Psi_L$, defined by equation (3) (paragraph 1.0.3), vanishes. Under
appropriate assumptions we show that if a component of the recurrent set is a cycle or an irrational 2-dimensional torus, the restriction of any limit measure to it is absolutely continuous with respect to the unique measure invariant under the flow generated by $b$ on the component (in the case of a torus, the measure is unique because our assumptions imply that the flow is ergodic). Moreover we give explicit formulas for the densities of these limit measures.

In the last step, we compute precise decay estimates for the eigenfunction and for that purpose, we use the Feynman-Kac formula for the eigenfunction.

2.2 Induction Principle for the localization of concentration

As stated in the main theorem, the selection of the recurrent set depends on the topological pressure only. On the other hand, the limit measures depend on the total jets of the field $b$ and the potential $c$ along the recurrent. The TP is not sufficient to determine the final support of the limit measures, because it involves only the first term in the expansion of the eigenvalue $\lambda_\epsilon$ in power of $\epsilon^{1/2}$. To obtain a more precise localization, all the expansion has to be used. This question is similar to the double-well potential problem: what are the coefficients of the limit measure when there are two recurrent sets where the TP is achieved (see [24])?

Remark

In this section we study the concentration of a limit measures on the critical points of the field $b$. Because $\Omega$ is not a gradient, we cannot use variational methods in our study (equation (4) can not be conjugated in general to a variational equations). Using the special Lyapunov function, constructed in [24], equation (4) is transformed into

$$\epsilon^2 \Delta_g v_\epsilon + \epsilon \theta(\Omega) v_\epsilon + c_\epsilon v_\epsilon = \epsilon \lambda_\epsilon v_\epsilon$$

$$c_\epsilon = \epsilon(c + \frac{\Delta_g \mathcal{L}}{2}) + \Psi_{\mathcal{L}}$$

where $c_\epsilon$ and $\Psi_{\mathcal{L}}$ have been computed in (3). This transformation is crucial in our analysis to obtain interesting results.

Weighting $v_\epsilon^2$ by $e^{-\frac{\epsilon}{\epsilon}}$ enables us to extract the main features of the limit measures. Another advantage of using the transformed equation is that the first order term tends to zero with $\epsilon$. Moreover, since the choice of the special Lyapunov function $\mathcal{L}$ is not unique, one would expect that the limits depend on this choice. In fact, it turns out that according to our construction of the Lyapunov function, the second order term of which satisfies the linearized Hamilton-Jacobi equation, the limit does not depend on this choice.
3 The case of critical points

3.1 Rate of convergence of the maximum points

Here we study the behavior of the maximum points of \( v_\epsilon = u_\epsilon \exp \frac{-L}{2} \). This is useful when we are going to normalize the eigenfunctions. Let \( \mathcal{M}_\epsilon \) be the set of maximum points of \( v_\epsilon \). \( \mathcal{R} \) denoted the recurrent set of \( b \).

**Lemma 1** There exists a constant \( C > 0 \) such that
\[
\sup \{ d(P, \mathcal{R}) | P \in \mathcal{M}_\epsilon \} \leq C \sqrt{\epsilon}.
\]

**Remark.** This situation is similar to the variational case [24] where the sequence \( P_\epsilon \) of global (also local) maximum points converges to a point of the recurrent set of the field. The rate of convergence depends on the order of the vanishing of \( \Psi_L \) on the recurrent sets.

**Proof:** The proof is an immediate consequence of the Maximum Principle. Indeed, at a maximum point (or local maximum) \( P \in \mathcal{M}_\epsilon \), \( \Delta_g v_\epsilon(P) \geq 0 \) and \( (\theta(\Omega) v_\epsilon)(P) = 0 \). Thus, using equation (4), because the sequence \( v_\epsilon \) is positive:
\[
c_\epsilon(P) \leq \epsilon \lambda_\epsilon.
\]

Because \( c_\epsilon = \epsilon (c + \frac{\Delta_g \mathcal{L}}{2}) + \Psi_L \), we obtain the following estimate
\[
0 \leq \Psi_L(P) \leq \epsilon (\lambda_\epsilon + \max |c + \frac{\Delta_g \mathcal{L}}{2}|).
\]

The definition of \( \Psi_L \) implies that there exists a constant \( C_1 > 0 \) such that
\[
\Psi_L(P) \geq \frac{1}{C_1} d(P, \mathcal{R})^2.
\]

Hence for some constant \( C \) and all \( P \in \mathcal{M}_\epsilon \)
\[
d(P, \mathcal{R})^2 \leq C \epsilon.
\]

3.2 Weak limits of the eigenfunctions \( w_\epsilon \): case of points

Let \((U, x_1, \ldots, x_m)\) be a coordinate system at \( P \), as defined in section 1.1(I). The blown up function \( w_\epsilon \) is defined on the subset \( \frac{1}{\sqrt{\epsilon}} x_1 \times \ldots \times x_m(U) \) of \( \mathbb{R}^m \) by
\[
w_\epsilon(x) = \frac{v_\epsilon(\sqrt{\epsilon} x)}{\bar{v}_\epsilon},
\]
where \( x = (x_1, \ldots, x_m) \). As \( \epsilon \) tends to 0 the behavior of \( w_\epsilon \) is described in the following theorem where \( \Delta_E = -\sum_{n=1}^{m} \frac{\partial^2}{\partial x_n^2} \).
Theorem 2 Let $P$ be a critical point of $b$ and let $(U, x_1, ..., x_m)$ be a normal coordinate system centered at $P$, as in section (I) of (1.1).

- Any weak limit $w$ of $w_\varepsilon$ as $\varepsilon \to 0$, is in $C^\infty$ and is a solution of the equation

$$\Delta_E w + \sum_{i,j=1}^m \Omega_{ij}(P) x_j \frac{\partial w}{\partial x_i} + [c(P) + \frac{\Delta_E L}{2}(P) + \psi_2(x)]w = \lambda w, \quad (8)$$

$$0 < w \leq 1,$$

where $\sum_{i,j=1}^m \Omega_{ij}(P) x_j \frac{\partial}{\partial x_i}$ is the linear part of the field $\Omega$ at $P$, $\psi_2(x)$ is the quadratic form representing the terms of order two in the Taylor development of $\Psi_L$ at $P$ and $\lambda \geq 0$ is equal to the topological pressure:

$$\Pi(P) = c(P) + \frac{\Delta_E L}{2}(P) + \sum \{\max(0, \text{Re}\sigma(\Omega(P))) | \sigma(\Omega(P)) \text{ eigenvalue of } \Omega(P)\},$$

where the sum in the right hand-side is the sum of all real parts of the eigenvalues of the matrix $(\Omega_{ij}(P)|1 \leq i, j \leq m)$, each counted according to its multiplicity.

- There exists a sequence $w_{\varepsilon_n}$ such that $\varepsilon_n \to 0$ as $n$ goes to infinity, which converges to $w$ in the $C^\infty$ topology.

- Either $w$ is identically zero or $0 < w \leq 1$. In addition, if the sequence is maximally charging at the point $P$ then $w(P) = 1$.

Proof. $w_\varepsilon$ satisfies the equation:

$$\Delta_g w_\varepsilon(x) + \sum_{i,j=1}^m \Omega^i_j(x, \sqrt{\varepsilon}) \frac{\partial w_\varepsilon(x)}{\partial x^i} + \left[ \frac{\Psi_L(x, \sqrt{\varepsilon})}{\varepsilon} + c + \frac{\Delta_g L(x)}{2} \right] w_\varepsilon(x) = \lambda w_\varepsilon(x) \quad (9)$$

where $g_\varepsilon(x) = g(\sqrt{\varepsilon}x)$. The functions $\hat{\Omega}$ and $\hat{\Psi}_L$ of the variables $(x, \varepsilon)$ defined for $1 \leq i \leq m$ by

$$\hat{\Omega}^i(x, \sqrt{\varepsilon}) = \frac{\Omega^i_j(\sqrt{\varepsilon}x)}{\sqrt{\varepsilon}}, \text{ for } \varepsilon > 0$$

$$= \sum_{j=1}^m \Omega^i_j(P)x_j, \text{ for } \varepsilon = 0$$

and

$$\hat{\Psi}_L(x, \sqrt{\varepsilon}) = \frac{\Psi_L(\sqrt{\varepsilon}x)}{\varepsilon} \text{ for } \varepsilon > 0$$

$$\hat{\Psi}_L(x, 0) = \psi_2(x) = \frac{1}{2} < A(P)x, x >_{\mathbb{R}^m},$$
where \(< A(P)x, x >_{\mathbb{R}^m}\) is the initial term of the Taylor expansion of \(\Psi_L\) at \(P\) are regular by assumptions on \(b\) and \(L\) at the point \(P\). Let us call \(L_\varepsilon\) the operator

\[
L_\varepsilon = \Delta g_\varepsilon + \sum_{i,j=1}^m \hat{\Omega}_j^i(x, \sqrt{\varepsilon}) \frac{\partial}{\partial x^i} + \hat{\Psi}_L(x, \sqrt{\varepsilon}) + c + \frac{\Delta g_\varepsilon L(x)}{2}.
\]

Because the second order operator \(L_\varepsilon\) is uniformly elliptic for \(\varepsilon \in [0, 1]\) (0 included!) on every compact set and its coefficients are \(C^\infty\) (in the variables \(x\) and \(\sqrt{\varepsilon}\)), classical interior elliptic estimates (see [21]) imply that \(w_\varepsilon\) is bounded on every compact set in the \(C^\infty\) topology uniformly in \(\varepsilon \in [0, 1]\), \(w_\varepsilon\) being bounded by 1. Ascoli-Arzela’s theorem implies \(w\) is the limit of a sequence \(\{w_\varepsilon| n \in \mathbb{N}, \varepsilon_n\}, \varepsilon_n \to 0\), which converges to \(w\) in the \(C^\infty\) topology. \(w\) is a classical solution of the elliptic equation

\[
\Delta_E w(x) + \sum_{i,j=1}^m B_{ij}(P)x_j \frac{\partial w}{\partial x_i} + [\psi_2(x) + (c + \Delta_E L/2)(0)]w(x) = \lambda w(x) \text{ on } \mathbb{R}^m.
\]

We want to determine the function \(w\). With that goal in mind we shall transform equation (8) into a well known one. Set

\[
z = w \exp \frac{< A(P)x, x >_{\mathbb{R}^m}}{2}.
\]

Then

\[
\Delta_E z + \sum_{i,j=1}^m (\Omega_j^i(P) + A_j^i(P))x_j \frac{\partial z}{\partial x_i} = [\lambda - c(0) - tr A]z
\]

If \(\sum_{i,j=1}^m B_{ij}(P)x_j \frac{\partial}{\partial x_i}\) denotes the linear part of \(b\) at \(P\) in the coordinate system \((U, x_1, ..., x_n)\):

\[
\Omega_j^i(P) + A_j^i(P) = B_j^i(P)
\]

The operator \(\Delta_E + \sum_{i,j=1}^m B_j^i(P)x_j \frac{\partial}{\partial x_i}\) is well known: it is the celebrated Ornstein-Uhlenbeck operator. \(z\) is a solution of

\[
\Delta_E z + \sum_{i,j=1}^m B_j^i(P)x_j \frac{\partial z}{\partial x_i} = [\lambda - c(0) - tr A]z
\]

If \(\sigma_1, \sigma_2, ..., \sigma_m\) are the eigenvalues of \(B(P)\), each appearing a number of times equal to its multiplicity, using the result of [32], we have

\[
\lambda = \Pi(P) = c(0) + Tr A - \sum_{n=1}^m \min[0, \text{Re}\sigma_n],
\]

equivalently

\[
\lambda - c(0) - tr A = -Tr B^s.
\]
3.3 The Ornstein-Uhlenbeck model

Clearly the function \( \zeta : \mathbb{R} \times \mathbb{R}^m \to \mathbb{R} \), \( \zeta(t, x) = z(x) \exp(t \text{Tr} B^*) \) is a solution of the parabolic equation

\[
\frac{\partial \zeta}{\partial t} + \Delta E \zeta + \sum_{ij=1}^m B^i_j(P) x_j \frac{\partial \zeta}{\partial x_i} = 0
\]  

(12)

We are going to show that Kolmogorov’s integral

\[
T_t(z)(x) = \frac{1}{(4\pi)^{m/2}(\det Q_t)^{1/2}} \int_{\mathbb{R}^m} e^{-<Q_t^{-1}(e^{-tB} x - y),(e^{-tB} x - y)>_{\mathbb{R}^m}/4} z(y) dy,
\]

\[
= \frac{1}{(4\pi)^{m/2}(\det Q_t)^{1/2}} \int_{\mathbb{R}^m} e^{-<Q_t^{-1}y,y>/4} z(e^tB x - y) dy
\]

is a \( C^\infty \) function satisfying the equation (12). Here

\[
Q_t = \int_0^t e^{-sB} e^{-sB^*} ds.
\]

Because \( z = w \exp \frac{<A(P)x,x>_{\mathbb{R}^m}}{2} \),

\[
T_t(z)(x) = \frac{1}{(4\pi)^{m/2}(\det Q_t)^{1/2}} \int_{\mathbb{R}^m} w(y) e^{-q(x,y,t)} dy
\]

(13)

where

\[
q(x, y, t) = \frac{1}{4} < Q_t^{-1}(e^{-tB} x - y), (e^{-tB} x - y) >_{\mathbb{R}^m} - \frac{A(P)y, y >_{\mathbb{R}^m}}{2},
\]

\[
q(x, y, t) = -\frac{1}{4} < U_t x, x >_{\mathbb{R}^m} + \frac{1}{4} ||R_{s,t}y_s - P_{s,t}x_s||_{\mathbb{R}^m}^2 + \frac{1}{4} ||R_{u,t}y_u - P_{u,t}y_u||_{\mathbb{R}^m}^2.
\]

(14)

where \( R_{s,t}, R_{u,t} \) are the unique positive definite operators such that:

\[
R_{s,t}^2 = Q_{s,t}^{-1} - 2A_s, \quad R_{u,t}^2 = Q_{u,t}^{-1} - 2A_u,
\]

\[
P_{s,t} = R_{s,t}^{-1}Q_{s,t}^{-1} e^{-B_s}, \quad P_{u,t} = R_{u,t}^{-1}Q_{u,t}^{-1} e^{-B_u},
\]

\[
U_t = U_{s,t} \oplus U_{u,t}
\]

where

\[
U_{s,t} = e^{-tB_s^*} (Q_{s,t}^{-1} - Q_{s,t}^{-1} R_{s,t}^{-2} Q_{s,t}^{-1}) e^{-tB_s}, \quad U_{u,t} = e^{-tB_u^*} (Q_{u,t}^{-1} - Q_{u,t}^{-1} R_{u,t}^{-2} Q_{u,t}^{-1}) e^{-tB_u}
\]

\( U_{s,t} \) is negative definite and \( U_{u,t} \) positive definite.

\[
T_t(z)(x) = \frac{e^{\frac{1}{4} < U_t x, x >_{\mathbb{R}^m}}}{(4\pi)^{m/2}(\det Q_t)^{1/2}} \int_{\mathbb{R}^m} w(y) \exp \left( -\frac{1}{4} ||R_{s,t}y_s - P_{s,t}x_s||_{\mathbb{R}^m}^2 - \frac{1}{4} ||R_{u,t}y_u - P_{u,t}y_u||_{\mathbb{R}^m}^2 \right) dy
\]
Lemma 2  The operators just defined have the following properties:

(i) For small \( t > 0 \):

1. \( Q_{s,t} = t \left[ \text{Id}_s - t \left( \frac{B_s + B^*_s}{2} \right) + O(t^2) \right] \) and \( Q^{-1}_{s,t} = \frac{1}{t} \text{Id}_s + \frac{B_s + B^*_s}{2} + O(t) \).

2. \( Q_{u,t} = t \left[ \text{Id}_u - t \left( \frac{B_u + B^*_u}{2} \right) + O(t^2) \right] \) and \( Q^{-1}_{u,t} = \frac{1}{t} \text{Id}_u + \frac{B_u + B^*_u}{2} + O(t) \).

3. \( R_{s,t}^2 = \frac{1}{t^2} \text{Id}_s + \frac{B_s + B^*_s}{2} - 2A_s + O(t) \) and \( R_{s,t}^{-2} = t \left[ \text{Id}_s - t \left( \frac{B_s + B^*_s}{2} - 2A_s \right) + O(t^2) \right] \)

4. \( R_{u,t}^2 = \frac{1}{t^2} \text{Id}_u + \frac{B_u + B^*_u}{2} - 2A_u + O(t) \) and \( R_{u,t}^{-2} = t \left[ \text{Id}_u - t \left( \frac{B_u + B^*_u}{2} - 2A_u \right) + O(t^2) \right] \)

5. \( U_{s,t} = -2A_s + O(t) \) and \( U_{u,t} = -2A_u + O(t) \).

6. \( R_{s,t}^{-1} Q_{s,t} e^{-tB_s} = \text{Id}_s + O(t) \) and \( R_{u,t}^{-2} Q_{u,t}^{-1} e^{-tB_u} = \text{Id}_u + O(t) \)

7. \( \det Q_t = t^{m-1} (1 + O(t)) \).

(ii) When \( t \to +\infty \):

1. \( Q_{s,t} \to \infty \) and \( Q_{u,t} \to \int_0^{+\infty} e^{-sB_u} e^{-sB^*_u} ds \)

2. \( R_{s,t} \to R_{s,\infty} = \sqrt{-2A_s} \) and \( R_{u,t} \to R_{u,\infty} = \sqrt{Q^{-1}_{u,\infty} - 2A_u} > 0 \)

3. \( Q_{s,t}^{-1} e^{-tB_s} \to 0 \) and hence \( P_{s,t} = R_{s,t}^{-2} Q_{s,t}^{-1} e^{-tB_s} \to 0, U_{s,t} \to \left( \int_0^{+\infty} e^{tB_s} e^{tB^*_s} d\tau \right)^{-1} \)

4. \( Q_{u,t} \to Q_{u,\infty} = \int_0^{+\infty} e^{-tB_u} e^{-tB^*_u} dt, P_{u,t} = R_{u,t}^{-2} Q_{u,t}^{-1} e^{-tB_u} \to 0, U_{u,t} \to 0. \)

(iii) \( e^{2tTrB_s} \det Q_{s,t} \to \det \int_0^{+\infty} e^{tB_s} e^{tB^*_s} d\tau \)

**Proof.** Most of the statements of the lemma are trivial. Let us prove (ii) 3).

\[
\left( Q_{s,t}^{-1} e^{-tB_s} \right)^{-1} = e^{tB_s} Q_{s,t} = e^{tB_s} \int_0^t e^{-\tau B_s} e^{-\tau B^*_s} d\tau = \left( \int_0^t e^{(t-\tau)B_s} e^{(t-\tau)B^*_s} d\tau \right) e^{-tB^*_s} = \left( \int_0^t e^{\sigma B_s} e^{\sigma B^*_s} d\sigma \right) e^{-tB^*_s}.
\]

Hence \( e^{tB_s} Q_{s,t} \to \infty, Q_{s,t}^{-1} e^{-tB_s} \to 0. \) We now compute

\[
U_{s,t} = e^{-tB^*_s} \left( Q_{s,t}^{-1} - Q_{s,t}^{-1} R_{s,t}^{-2} Q_{s,t}^{-1} \right) e^{-tB_s} = e^{-tB^*_s} Q_{s,t}^{-1} e^{-tB_s} - e^{-tB^*_s} Q_{s,t}^{-1} R_{s,t}^{-2} Q_{s,t}^{-1} e^{-tB_s}.
\]

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Hence 
\[ e^{-tB_s} Q_{s,t}^{-1} = \left( Q_{s,t}^{-1} e^{-tB_s} \right)^* \to 0 \]
\[ e^{-tB_s} Q_{s,t}^{-1} R_{s,t}^{-2} Q_{s,t}^{-1} e^{-tB_s} \to 0. \]

Moreover
\[ e^{-tB_s} Q_{s,t}^{-1} e^{-tB_s} = \left( e^{tB_s} Q_{s,t} e^{tB_s} \right)^{-1} e^{tB_s} Q_{s,t} e^{tB_s} = e^{tB_s} \int_0^t e^{-\tau B_s} e^{-\tau B_s^*} d\tau e^{tB_s}. \]

Thus
\[ e^{tB_s} Q_{s,t} e^{tB_s} = \int_0^t e^{(t-\tau) B_s} e^{(t-\tau) B_s^*} d\tau = \int_0^t e^{\sigma B_s} e^{\sigma B_s^*} d\sigma. \]

Hence as \( t \to +\infty \),
\[ e^{-tB_s} Q_{s,t}^{-1} e^{-tB_s} \to \left( \int_0^{+\infty} e^{\sigma B_s} e^{\sigma B_s^*} d\sigma \right)^{-1}. \]

and
\[ U_{s,t} \to \left( \int_0^{+\infty} e^{\tau B_s} e^{\tau B_s^*} d\tau \right)^{-1}. \]

To prove (ii) 4) note that
\[ U_{u,t} = e^{-tB_u^*} (Q_{u,t}^{-1} - Q_{u,t}^{-1} R_{u,t}^{-2} Q_{u,t}^{-1}) e^{-tB_u}. \]

Because
\[ Q_{u,t}^{-1} - Q_{u,t}^{-1} R_{u,t}^{-2} Q_{u,t}^{-1} \to Q_{u,\infty}^{-1} - Q_{u,\infty}^{-1} R_{u,\infty}^{-2} Q_{u,\infty}^{-1}, \]

\( U_{u,t} \) tends to 0.

To prove (iii) note that
\[ e^{tB_s} Q_{s,t} e^{tB_s} \to \int_0^{+\infty} e^{\tau B_s} e^{\tau B_s^*} d\tau. \]

Hence
\[ det e^{tB_s} \det Q_{s,t} \det e^{tB_s^*} \to \det \int_0^{+\infty} e^{\tau B_s} e^{\tau B_s^*} d\tau. \]

But
\[ det e^{tB_s} = det e^{tB_s^*} = e^{TrB_s}. \]
So as \( t \to +\infty \),

\[
e^{2t \text{Tr} B_s} \det Q_{s,t} \to \det \int_0^{+\infty} e^{\tau B_s} e^{\tau B_s^*} d\tau
\]

The formulas (13), (14) imply that \( T_t(z) \) is a \( C^\infty \) function of \( x \) for \( t > 0 \). The integral

\[
e^{-t \text{Tr} B_s} \oint_{\mathbb{R}^m} w(y) \exp -\frac{1}{4} \left\{ ||R_{s,t} y_s - P_{s,t} x_s||_{\mathbb{R}^m}^2 + ||R_{u,t} y_u - P_{u,t} y_u||_{\mathbb{R}^m}^2 \right\} dy (15)
\]

is bounded by \( \sup w \) for all \( t > 0 \), because by Lemma 2, (ii) and (iii) 2)

\[
e^{t \text{Tr} B_s} \sqrt{\det Q_{s,t}} \to \sqrt{\int_0^{+\infty} e^{\tau B_s} e^{\tau B_s^*} d\tau} (16)
\]

and \( R_{s,t} \to \sqrt{-2A_s} \) and \( R_{u,t} \to R_{u,\infty} = \sqrt{Q_{u,\infty}^{-1} - 2A_u} \). Hence \( |T_t(z)(x)| \) is bounded by \( C_t \exp -\frac{1}{4} <u_{s,t} x_s, x_s>_{\mathbb{R}^m} - \frac{1}{4} <u_{u,\infty} - 2A_u>_{\mathbb{R}^m} dy \).

These properties imply by Tikhonov’s theorem [13]:

**Lemma 3** for all \( t > 0 \), \( T_t(z) = e^{-t \text{Tr} B_s} z \).

**Proof.** It is sufficient to prove that \( T_t(z) \to z \) and that \( C_t \to 1 \) as \( t \to 0 \). Using Lemma (2) and the representation (15) we can show that:

\[
T_t(z)(x) = \exp \frac{1+O(t)}{2} <Ax, x>_{\mathbb{R}^m} \times
\]

\[
\int_{\mathbb{R}^m} w(\eta) (1 + O(t)) e^{-\frac{1+O(t)}{2} (||\eta||_{\mathbb{R}^m}^2 + ||\eta||_{\mathbb{R}^m}^2)} d\eta.
\]

Because \( w \) is bounded at \( \infty \), we see that \( T_t(z)e^{-\frac{1}{2}<Ax, x>_{\mathbb{R}^m}} \) and \( w e^{-t \text{Tr} B_s} \) tend to \( w \) as \( t \to 0 \), uniformly in \( x \) and that \( C_t \to 1 \). The functions \( T_t(z)e^{-\frac{1}{2}<Ax, x>_{\mathbb{R}^m}} \) and \( w \) are both bounded by quadratic exponentials and satisfy the same parabolic equation

\[
\Delta E h + \sum_{i,j=1}^m \Omega_j^i (P) x_j \frac{\partial h}{\partial x_i} + [\psi_2 + (c + \Delta E \mathcal{L}/2)(0) - \lambda] h + \frac{\partial h}{\partial t} = 0.
\]

and both functions have the same limits as \( t \to 0 \), Tikhonov’s theorem [13] implies

\[
T_t(z) = e^{-t \text{Tr} B_s} z.
\]
Lemma 4 The solution $z$ does not depend on the unstable variables $x_u$ and is given explicitly for $x = (x_s, x_u) \in \mathbb{R}^m$ by

$$z(x) = C \exp \left( -\frac{1}{4} < \left( \int_0^{+\infty} e^{tB_s}e^{tB_s^*}dt \right)^{-1} x_s, x_s >_{\mathbb{R}^m} \right)$$

where $C$ is a constant.

**Proof.** Using lemma (3), Kolmogorov’s formula (29) gives:

$$z(x) = \frac{e^{-tTrB_s + \frac{1}{4} < U_t x, x >_{\mathbb{R}^m}}}{(4\pi)^{m/2}(detQ_t)^{1/2}} \int_{\mathbb{R}^m} w(\eta + R^{-1}_t P_t x) e^{-\frac{1}{4} ||R_s, t\eta||^2_{\mathbb{R}^m} - \frac{1}{4} ||R_{u, t}\eta||^2_{\mathbb{R}^m}} d\eta \quad (17)$$

Letting $t \to +\infty$ in the expression (17) above, using the preceding estimates in Lemma (2) and the fact that $w$ is bounded at $\infty$, we see that:

$$z(x) = \frac{\exp \left[ -\frac{1}{4} \left( \int_0^{+\infty} e^{tB_s}e^{tB_s^*}dt \right)^{-1} x_s, x_s >_{\mathbb{R}^m} \right]}{(4\pi)^{m/2} \sqrt{det \int_0^{+\infty} e^{tB_s}e^{tB_s^*}dt}} \times \int_{\mathbb{R}^m} w(\eta) e^{-\frac{1}{4} ||R_s, \eta||^2_{\mathbb{R}^m} - \frac{1}{4} ||R_{u, \eta}||^2_{\mathbb{R}^m}} d\eta \quad (17)$$

Let us summarize in the following theorem, the results obtained so far

**Theorem 3** Let $(V_m, g)$ be a Riemannian compact manifold, and $b$ be a Morse-Smale vector field the recurrent set $\mathcal{R}$ of which is a finite number of points. Let $\mathcal{L}$ be a special Lyapunov function. The limits $\mu$ of the probability measures associated to the eigenfunction $u_\epsilon$ of $\mathcal{L}_\epsilon$, defined by

$$\frac{e^{-\mathcal{L}_\epsilon / \epsilon u_\epsilon^2 dV_g}}{\int_{V_m} e^{-\mathcal{L}_\epsilon / \epsilon u_\epsilon^2 dV_g}}$$

are of the form

$$\mu = \sum_{P \in S_{tp}} c_P \delta_P,$$

where the set $\mathcal{R}_{tp}$ is the subset of $\mathcal{R}$ where the topological pressure is achieved and $\sum_{P \in \mathcal{R}_{tp}} c_P = 1, c_P \geq 0.$
1. Wherever the topological pressure is achieved, the solution \( w \) of the blow up equation (8) satisfies,

\[
\Delta E w + (\Omega_{ij} x^i, \nabla_j w) + [\Psi_L(x) + (c + \Delta E L/2)(0)] w = \lambda w \text{ on } \mathbb{R}^m
\]

\[
0 \leq w \leq 1.
\]

\( w \) is \( C^\infty \). It attains its maximum and decays quadratically exponentially fast at \( \infty \): there exists a constant \( C > 0 \) such that

\[
\forall x \in \mathbb{R}^m, \ w(x) \leq e^{-C||x||_{\mathbb{R}^m}}.
\]

2. There exists a constant \( C(b, g) \) such that

\[
\lim_{\epsilon \to 0} \epsilon^{m/2} v_\epsilon^2 = C(b, g),
\]

where \( v_\epsilon = \sup_{P \in V} v_\epsilon(P) \).

3. The coefficient \( c_P \) can be computed from \( C(b, g) \). The function \( w_P \) is equal to the blow up solution \( w \) at the point \( P \) and the concentration coefficient is computed using the modulating factor \( \gamma_P \) and is given

\[
c_P = C(b, g) \gamma_P^2 \int_{\mathbb{R}^m} w_P^2.
\]

4. Let us call \( \mathcal{CR} \) the set of recurrent points, which are charged (\( \gamma_P > 0 \) for all \( P \in \mathcal{CR} \)), then

\[
C(b, g) = \frac{1}{\sum_{P \in \mathcal{CR}} \gamma_P^2 \int_{\mathbb{R}^m} w_P^2}.
\]

**Proof.** The proof is a consequence of Theorem 2 and the previous lemmas. Using these results, the proofs of the statements of theorem 3 follow the same steps as in theorem 4 of [24]. Anyway, the proof of the parts 2-3-4 will be given in the section 4 about limit cycles.

### 4 Concentration near limit cycles

When the recurrent set of the MS vector field \( \Omega \) contains limit cycles, the limits of the probability measures associated to the eigenfunctions can have components located on the limit cycles. Thus it is relevant to study the restrictions of the limit measures to the cycles. We shall prove here that they are absolutely continuous with respect to the length induced by the metric \( g \) along the cycle.
Moreover, we will show that the limit measures are concentrated on the cycles or critical points where the topological pressure is attained. Actually if some limit cycles are charged, then no critical point is charged. The reason for this is: Once a limit cycle is charged, then the speed with which the maximum of the eigenfunction tends to infinity when $\epsilon$ goes to zero is determined by the local behavior near the cycle.

4.1 Statement of the main results

This section is devoted to the proof of the second part of theorem 1 and auxiliary propositions.

**Proposition 1** Let $S$ be a limit cycle, the restriction of any limit of the eigenfunction measures to a sufficiently small tubular neighborhood of $S$ is carried by $S$ and has the following form $c_S f \delta_S$, where $f: S \rightarrow \mathbb{R}$ is the unique normalized (maximum equal one) periodic solution of

$$\frac{\partial f}{\partial \theta} + c(S(\theta))f = \lambda f(\theta) \text{ on } S$$

where

$$\lambda = \frac{1}{T_S} \int_0^{T_S} c(S(\theta))d\theta.$$

c$_S$ is the concentration coefficient, $S(\theta)$ the parametrization of a cycle and $T_S$ the minimal period of $S$.

**Remarks.**

The next two propositions describe the behavior of the renormalized sequence of eigenfunctions near the limit cycles. The notations are the same as before, $\sup_{V_n} v_\epsilon = v(P_\epsilon) = \bar{v}_\epsilon$ and $P_\epsilon$ is a maximum point of $v_\epsilon$. Finally $\Delta_{E}^{m-1}$ denotes opposite the Laplacian operator in $\mathbb{R}^{m-1}$.

**Lemma 5**

1. If $S$ is charged, it contains limits of sequences $P_{\epsilon_n}$, where $\epsilon_n$ tends to zero (see lemma 1).

2. Let $P_{\epsilon_n}, n \in \mathbb{N}$, where $\epsilon_n$ tends to zero, be a sequence of maximum points ($v_{\epsilon_n}(P_{\epsilon_n}) = \sup_{V_n} v_{\epsilon_n}$), which converges to a point $P_0 \in S$. Then by Lemma 1, since $\frac{d(P_{\epsilon_n}, P_0)}{\sqrt{\epsilon_n}} \leq Cte$, passing to a subsequence if necessary, we assume that $\frac{d(P_{\epsilon_n}, P_0)}{\sqrt{\epsilon_n}}$ converges to a finite limit and $\frac{1}{\sqrt{\epsilon_n}}(\delta_{P_{\epsilon_n}} - \delta_{P_0})$ to $\Theta(T)$, where $T \in T_{P_0}(V)$ and $\Theta(T)$ the Lie derivative associated to $T$.

The proof is a consequence of lemma 1.
Proposition 2 Any sequence of $\epsilon$ converging to zero contains a sub-sequence $\epsilon_n$ such that the blow-up sequence defined by

$$w_{\epsilon_n}(x', \theta) = \frac{v_{\epsilon_n}(\sqrt{\epsilon_n}x', \theta)}{v_{\epsilon_n}}$$

converges uniformly to a function $w$ on every compact of $\mathbb{R}^{m-1} \times \mathbb{R}$. $w$ is a periodic solution of period $T_S$ in $\theta$ of equation:

$$\Delta_{E}^{m-1} w + \sum_{i,j=1}^{m-1} \Omega_{ij} x_i \frac{\partial w}{\partial x^j} + \frac{\partial w}{\partial \theta} + (c(0, \theta) + \frac{\Delta g L(0, \theta)}{2} + \psi_2(x')) w = \lambda w \quad (18)$$

where $\psi_2(x')$ are the terms of order 2 in the Taylor expansion of the Lyapunov function $L$ along $S$. $1 \geq w > 0$, and the maximum 1 is attained. In the equation (18), $\lambda > 0$ equals the topological pressure at $S$, that is

$$\lambda = \langle c_0 \rangle >_S -Tr B^*.$$ 

Finally, using the Orthogonality Assumption with the same notations as above, we have

Proposition 3 Under the orthogonal Assumptions along the cycle, in a co-ordinate system defined in 1.1-iii, $x = (x_u, x_s, \theta)$, where $x_u$ represents the unstable direction, $x_s$ the stable direction and $\theta$ a parametrization of the cycle, the solution $w$ is given by

$$w(x) = z(x_u, x_s) f(\theta),$$

where $z$ is solution of

$$\Delta_E^{m-1} z + \sum_{i,j} A_{ij}^s \frac{\partial z}{\partial x^i} + \left[ \frac{\Delta g L(P_0)}{2} + \bar{Q}(x') \right] z = \lambda z \text{ on } \mathbb{R}^{n-1},$$

$$1 \geq z > 0,$$

and $f$ is a periodic solution along the cycle of the equation

$$\frac{\partial f}{\partial \theta} + c(0, \theta) f = \langle c_0 \rangle f,$$

where $\langle c_0 \rangle$ is average of $c$ along the cycle.

Proposition 4 given below gives the final information necessary to characterize the limit measure on the cycle and the value of the concentration coefficients.
Theorem 4  
1. When a limit cycle of $\Omega$ is charged, for any charging sequence there exists a subsequence, $\{\varepsilon_n| n \in \mathbb{N}\}$ and a strictly positive constant $C(\mu) > 0$, such that 

$$
\lim_{n \to \infty} \varepsilon_n^{m/2-1} \bar{v}_n^2 = C(\mu). 
$$

(19) 

2. Suppose that there exists a subsequence converging to a measure $\mu$ such that only the cycles $S_1, \ldots, S_p$ are charged by this measure. If the cycles are of minimal period $T_1, \ldots, T_q$, then 

$$
C(\mu) = \frac{1}{\sum_{r=1}^p \gamma_r^2 \int_{\mathbb{R}^{n-1}} z^2_r(x') dx' \int_0^{T_r} f^2_r(\theta) d\theta}, 
$$

where $\gamma_r$ is the modulating coefficients. 

3. The concentration coefficients $c_S$ along a charged cycle $S$ is given by:

$$
c_S = \lim_{n \to \infty} \int_{T_S(\delta)} v^2_{\varepsilon_n} = C(\mu) \gamma_S^2 \int_0^{T_S} f^2_S(\theta) d\theta \int_{\mathbb{R}^{m-1}} z^2_S(x') dx', 
$$

where $T_S(\delta)$ is a tubular neighborhood of the cycle, $T_S$ the minimal period and the functions $f_S, z_S$ come from the blow-up analysis of the previous results (theorem 4 and proposition 3).

4.2 Proofs of the theorems.

Recall that the normalized eigenfunction $v_\varepsilon$ satisfies

$$
\varepsilon^2 \Delta_g v_\varepsilon + \varepsilon(\Omega, \nabla v_\varepsilon) + c_\varepsilon v_\varepsilon = \varepsilon \lambda_\varepsilon v_\varepsilon, 
$$

(20) 

$$
\int_{V_m} v^2_\varepsilon dV_g = 1
$$

Proof proposition 2 and 3.

Consider a coordinate system $x = (x', \theta)$ on the universal covering of a tubular neighborhood of the cycle $S$ as defined in (1.1) III. The blown-up function is defined by

$$
w_\varepsilon(x', \theta) = \frac{v_\varepsilon(\sqrt{\varepsilon} x', \theta)}{\bar{v}_\varepsilon}.
$$

Consider the equation

$$
\Delta^{m-1}_E w + \sum_{i,j=1}^{m-1} \Omega_j x^j \frac{\partial w}{\partial x^i} + \frac{\partial w}{\partial \theta} + (c(0, \theta) + \frac{\Delta_g L(0, \theta)}{2} + \psi_2(x')) w = \lambda w 
$$

(21)
Lemma 6 Any sequence of \(\epsilon\)'s converging to zero contains a sub-sequence \(\epsilon_n\) such that the blown-up sequence \(w_{\epsilon_n}\) converges to a function \(w \geq 0\) solution of equation (21) uniformly on any compact set \(K \times S^1\).

Proof. The blown-up metric is defined by

\[
g_\epsilon(x', \theta) = g(\sqrt{\epsilon}x', \theta).
\]

Define \(\Gamma^k_{\epsilon ij}\) by \(\Gamma^k_{\epsilon ij}(x', \theta) = \Gamma^k_{ij}(\sqrt{\epsilon}x', \theta)\) where the \(\Gamma^k_{ij}\) are the Christoffel symbols of the metric \(g\) in the coordinates \((x_1, ..., x_{m-1}, \theta)\). Note that the \(\Gamma^k_{\epsilon ij}\) are not the Christoffel symbols of the metric \(g_\epsilon\). As \(\epsilon\) goes to zero, the sequence \(g_\epsilon\) converges uniformly to the metric \(g_E = \sum_{n=1}^{m-1} dx^2_n + g^{\theta\theta}(0, \theta) d\theta^2\).

The sequence \(w_\epsilon\) satisfies the equation

\[
L_0w_\epsilon + \sqrt{\epsilon}L_1w_\epsilon = \lambda_\epsilon w_\epsilon
\]

with

\[
L_1 = D_1 + \sqrt{\epsilon}D_2
\]

where:

\[
L_0 = \Delta_E^{m-1} + \frac{\partial}{\partial \theta} + \sum_{i,j=1}^{m-1} \Omega^i_j x^j \frac{\partial}{\partial x^i} + c(0, \theta) + \frac{TrA}{2} + \left( \frac{B^{\theta\theta} A + AB}{2} - A^2 \right) x', x' > \mathbb{R}^{m-1}
\]

\(L_0\) is the limit of the blown-up operator

\[
\epsilon \Delta_g v_\epsilon = \Delta_E^{m-1} w_\epsilon + \sqrt{\epsilon}D_1(w_\epsilon) + \epsilon D_2(w_\epsilon)
\]

where

\[
-\Delta_E^{m-1} := \sum_{i,j=1}^{m-1} g^{ij}_\epsilon \frac{\partial^2}{\partial x_i \partial x_j},
\]

and

\[
D_1 := -\sum_{k=1}^{m-1} \left( \sum_{i,j=1}^{m-1} g^{ij}_\epsilon \Gamma^k_{\epsilon ij} + g^{\theta \theta}_\epsilon \Gamma^k_{\epsilon \theta \theta} \right) \frac{\partial}{\partial x_k} + 2 \sum_{i=1}^{m-1} g^{\theta i}_\epsilon \left( \frac{\partial^2}{\partial x_i \partial \theta} - \sum_{k=1}^{m-1} \Gamma^k_{\epsilon \theta i} \frac{\partial}{\partial x_k} \right)
\]

and

\[
D_2 := \left( g^{\theta \theta}_\epsilon \left( \frac{\partial^2}{\partial \theta \partial \theta} - \Gamma^\theta_{\epsilon \theta \theta} \frac{\partial}{\partial \theta} \right) - \sum_{i,j=1}^{m-1} g^{ij}_\epsilon \Gamma^\theta_{\epsilon ij} \frac{\partial}{\partial \theta} - 2 \sum_{i=1}^{m-1} g^{\theta i}_\epsilon \Gamma^\theta_{\epsilon \theta i} \frac{\partial}{\partial \theta} \right).
\]

Equation (22) is considered in the domain \(B^{m-1}(0, \frac{\delta}{\sqrt{\epsilon}}) \subset \mathbb{R}^{m-1} \times S^1\) and \(g_\epsilon\) converges to \(g_E\) uniformly on each compact set, where \(g_E = \sum_{i=1}^{m-1} dx^2_i + g^{\theta\theta}(0, \theta) d\theta^2\).
Limit equation. Any weak limit $w$ of $w_\varepsilon$ when $\varepsilon$ goes to zero, satisfies the equation
\[
\Delta_{E}^m w + \frac{\partial w}{\partial \theta} + \sum_{i,j=1}^{m-1} \Omega_{ij} x_j \frac{\partial w}{\partial x_i} + (c(0, \theta) + TrA+ \left( \frac{B^* A + AB}{2} - A^2 \right) x', x' >_{\mathbb{R}^m-1}) w = \lambda w \tag{23}
\]
where $0 \leq w \leq \text{ess sup } w = 1$ and $\lim_{\varepsilon \to 0} \lambda_\varepsilon = \lambda$. Actually, the sequence $w_\varepsilon$ converges in the $C^\infty$ topology on any compact set of $\mathbb{R}^{m-1} \times S^1$, as proved in the appendix.

We now proceed with the computation of an explicit expression of the function solution of equation (23). Let us introduce the simplified notations $c_0(\theta) = c(0, \theta)$. We replace the function $w(x', \theta)$ by $\tilde{w}(x', \theta)$,
\[
\tilde{w}(x', \theta) = w(x', \theta) \exp\left\{ \int_0^\theta \left[ c_0(t) - \frac{<c_0>}{T_S} \right] dt \right\},
\]
$\tilde{w}$ is bounded, periodic in $\theta$ and solution of
\[
\Delta_{E}^m \tilde{w} + \sum_{i,j=1}^{m-1} \Omega_{ij} x_j \frac{\partial \tilde{w}}{\partial x_i} + \frac{\partial \tilde{w}}{\partial \theta} + (\frac{TrA}{2} + <Ax', x'>) \tilde{w} = [\lambda - <c_0>] \tilde{w} = (-TrB_s) \tilde{w}.
\]
A final gauge transformation $\tilde{w} = ze^{-\frac{1}{4}<Ax', x'>}$ leads to the equation
\[
\Delta_{E}^m z + \sum_{i,j=1}^{m-1} (\Omega_{ij} + A_{ij}) x_j \frac{\partial z}{\partial x_i} + \frac{\partial z}{\partial \theta} = (-TrB_s) z.
\]
Going back to the original vector field $b$, we have
\[
\Delta_{E}^m z + \sum_{i,j=1}^{m-1} B_{ij} x_j \frac{\partial z}{\partial x_i} + \frac{\partial z}{\partial \theta} = (-TrB_s^*) z. \tag{24}
\]

Lemma 7 The function $z$ is given by Kolmogorov’s formula:
\[
z(x', \theta) = \frac{e^{-\theta TrB^*}}{(4\pi)^{m-1} \sqrt{\det Q_\theta}} \int_{\mathbb{R}^m-1} \tilde{w}(y, 0) e^{-\theta q(x', y', \theta)} dy \tag{25}
\]
where $Q_\theta = \int_0^\theta e^{-tB} e^{-tB^*} dt$ and
\[
q(x', y', \theta) = \frac{1}{4} <Q_\theta^{-1}(e^{-\theta B} x' - y'), (e^{-\theta B} x' - y') >_{\mathbb{R}^m-1} - \frac{1}{2} <Ay', y' >_{\mathbb{R}^m-1} \tag{26}
\]
Proof:

\[ z(x', \theta) = \tilde{w}(x', \theta) \exp \left( \frac{1}{2} < Ax', x' >_{\mathbb{R}^{m-1}} \right). \] \hfill (27)

Because \( \tilde{w} \) is bounded, for some constants \( C_1 > 0, C_2 > 0 \), for all \( x', \theta \),

\[ |z(x', \theta)| \leq C_1 e^{C_2 \|x'\|^2_{\mathbb{R}^{m-1}}}. \] \hfill (28)

Thus \( z \) belongs to the Tikhonov class and hence is entirely determined by its value for a given fixed \( \theta \) by Tikhonov’s theorem. Let us introduce the function:

\[ v(x', \theta) = e^{-\theta \text{tr} B_s/(4\pi)^{m/2} \sqrt{\det Q_\theta} \int_{\mathbb{R}^{m-1}} \tilde{w}(y', 0) e^{-q(x', y', \theta)} dy'}. \] \hfill (29)

Lemma 8 completes the proof of lemma 7 and shows that \( v = z \).

**Lemma 8**

1. \( v \) is well defined.
2. \( v \) is a solution of equation (24)
3. \( v(x', 0) = z(x', 0) \)
4. \( v \) belongs to Tikhonov’s class.

**Proof.** Let us estimate \( q(x, y, \theta) \). We have the decomposition \( \mathbb{R}^m = W^s \times W^u \times \mathbb{R} \), where \( W^s \) and \( W^u \) are the stable and unstable space respectively. If \( x' \in \mathbb{R}^{m-1} \), we denote by \( x_s \) (resp. \( x_u \)) the stable (resp unstable) components.

To this decomposition of \( \mathbb{R}^m \) corresponds the splitting of matrices:

\[
B = \begin{bmatrix}
B_s & 0 & 0 \\
0 & B_u & 0 \\
0 & 0 & 0
\end{bmatrix},
Q_\theta = \begin{bmatrix}
Q_{s, \theta} & 0 & 0 \\
0 & Q_{u, \theta} & 0 \\
0 & 0 & 0
\end{bmatrix},
R_\theta = \begin{bmatrix}
R_{s, \theta} & 0 & 0 \\
0 & R_{u, \theta} & 0 \\
0 & 0 & 0
\end{bmatrix},
\]

\[
P_\theta = \begin{bmatrix}
P_{s, \theta} & 0 & 0 \\
0 & P_{u, \theta} & 0 \\
0 & 0 & 0
\end{bmatrix},
U_\theta = \begin{bmatrix}
U_{s, \theta} & 0 & 0 \\
0 & U_{u, \theta} & 0 \\
0 & 0 & 0
\end{bmatrix},
A_\theta = \begin{bmatrix}
A_{s, \theta} & 0 & 0 \\
0 & A_{u, \theta} & 0 \\
0 & 0 & 0
\end{bmatrix},
\]

where

\[
Q_{s, \theta} = \int_0^\theta e^{-tB_s} e^{-tB_s^*} dt, Q_{u, \theta} = \int_0^\theta e^{-tB_u} e^{-tB_u^*} dt, A_s^{-1} = -\int_0^{+\infty} e^{tB_s} \Pi_s e^{tB_s^*} dt, (30)
\]

\[
A_u^{-1} = \int_0^{+\infty} e^{-tB_u} \Pi_u e^{-tB_u^*} dt, \quad (31)
\]

and \( \Pi_s, \Pi_u \) are positive-definite and >> \( 2I_{s} \) (resp. \( 2I_{u} \)) in the natural order on the symmetric operators. Hence \( A_u^{-1} >> 2Q_{u, \theta} \). This implies that \( Q_{u, \theta}^{-1} - 2A_u \)
is positive-definite and obviously so is $Q_{s,\theta}^{-1} - 2A_s$. Let $R_{s,\theta}$ and $R_{u,\theta}$ be the unique positive-definite symmetric operators such that $R_{s,\theta}^2 = Q_{s,\theta}^{-1} - 2A_s$ and $R_{u,\theta}^2 = Q_{u,\theta}^{-1} - 2A_u$. We now need the following operators to express $q$. Define

$$
P_{s,\theta} = R_{s,\theta}^{-1}Q_{s,\theta}^{-1}e^{-\theta B_s}, P_{u,\theta} = R_{u,\theta}^{-1}Q_{u,\theta}^{-1}e^{-\theta B_u}.
$$

Then

$$
q(x, y, \theta) = \frac{1}{4} \left[ <U_{s,\theta}x, x >_{\mathbb{R}^{m-1}} + <U_{u,\theta}x, x >_{\mathbb{R}^{m-1}} + ||R_{s,\theta}y - P_{s,\theta}x||_{\mathbb{R}^{m-1}}^2 + ||R_{u,\theta}y - P_{u,\theta}y||_{\mathbb{R}^{m-1}}^2 \right], \tag{32}
$$

where

$$
U_{s,\theta} = e^{-\theta B_s^*} \left( Q_{s,\theta}^{-1} - Q_{s,\theta}^{-1}R_{s,\theta}^{-2}Q_{s,\theta}^{-1} \right) e^{-\theta B_s}, U_{u,\theta} = e^{-\theta B_u^*} \left( Q_{u,\theta}^{-1} - Q_{u,\theta}^{-1}R_{u,\theta}^{-2}Q_{u,\theta}^{-1} \right) e^{-\theta B_u},
$$

then

$$
U_{s,\theta} = e^{-\theta B_s^*} Q_{s,\theta}^{-1} (Q_{s,\theta} - R_{s,\theta}^{-2}) Q_{s,\theta}^{-1} e^{-\theta B_s}.
$$

Because $R_{s,\theta}^2 >> Q_{s,\theta}^{-1}$, $Q_{u,\theta}^2 >> R_{u,\theta}^2$ and $U_{s,\theta}$ is positive definite. On the other hand

$$
U_{u,\theta} = e^{-\theta B_u^*} Q_{u,\theta}^{-1} (Q_{u,\theta} - R_{u,\theta}^{-2}) Q_{u,\theta}^{-1} e^{-\theta B_u}.
$$

Because $2A_u$ is positive definite, $Q_{u,\theta}^{-1} >> R_{u,\theta}^2$ and hence $R_{u,\theta}^{-2} >> Q_{u,\theta}$. So $U_{u,\theta}$ is negative definite.

Relation (32) shows that $v$ is well defined. Making the change of variables $y' \rightarrow \eta'$, $\eta' = y' - R_{\theta}^{-1}P_{\theta}x'$ in the integral of equation (29) we get that

$$
v(x', \theta) = \exp \left[ -\frac{\theta Tr B_s + \frac{1}{4} <U_{\theta}x', x' >_{\mathbb{R}^{m-1}}}{(4\pi)^{m-1} \sqrt{\det Q_{\theta}}} \right] \times
$$

$$
\int_{\mathbb{R}^{m-1}} \tilde{w}(\eta' + R_{\theta}^{-1}P_{\theta}x'), 0) e^{-\frac{1}{2}||R_{s,\theta}\eta'||^2_{\mathbb{R}^{m-1}} - \frac{1}{2}||R_{u,\theta}\eta'||^2_{\mathbb{R}^{m-1}}} d\eta'.
$$

Because $\tilde{w}$ is bounded, $v$ is in Tikhonov’s class. Now it can be checked that $v$ satisfies equation (24). Let us show that

$$
\lim_{\theta \rightarrow 0} v(x', \theta) = \z(x', 0).
$$

But first let us state a lemma:

**Lemma 9** The operators just defined have the following properties:

(i) For small $\theta > 0$:

1. $Q_{s,\theta} = \theta \left[ Id_s - \theta \left( \frac{B_s + B_s^*}{2} \right) + O(\theta^2) \right]$ and $Q_{u,\theta}^{-1} = \frac{1}{\theta} Id_s + \frac{B_s + B_s^*}{2} + O(\theta)$. 

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2. \( Q_{u,\theta} = \theta \left[ I_{du} - \theta \left( \frac{B_u + B_u^*}{2} \right) + O(\theta^2) \right] \) and \( Q_{u,\theta}^{-1} = \frac{1}{\theta} I_{du} + \frac{B_u + B_u^*}{2} + O(\theta) \).

3. \( R_{s,\theta}^2 = \frac{1}{\theta} I_{ds} + \frac{B_s + B_s^*}{2} - 2A_s + O(\theta) \) and \( R_{s,\theta}^{-2} = \theta \left[ I_{ds} - \theta \left( \frac{B_s + B_s^*}{2} - 2A_s \right) + O(\theta^2) \right] \).

4. \( R_{u,\theta}^2 = \frac{1}{\theta} I_{du} + \frac{B_u + B_u^*}{2} - 2A_u + O(\theta) \) and \( R_{u,\theta}^{-2} = \theta \left[ I_{du} - \theta \left( \frac{B_u + B_u^*}{2} - 2A_u \right) + O(\theta^2) \right] \).

5. \( U_{s,\theta} = -2A_s + O(\theta) \) and \( U_{u,\theta} = -2A_u + O(\theta) \).

6. \( R_{s,\theta}^{-2} Q_{s,\theta}^{-1} e^{-\theta B_s} = I_{ds} + O(\theta) \) and \( R_{u,\theta}^{-2} Q_{u,\theta}^{-1} e^{-\theta B_u} = I_{du} + O(\theta) \).

7. \( \det Q_\theta = \theta^{m-1}(1 + O(\theta)) \).

(ii) When \( \theta \to +\infty \):

1) \( Q_{s,\theta} \to \infty \) and \( Q_{u,\theta} \to \int_0^{+\infty} e^{-sB_u} e^{-sB_s^*} ds \)

2) \( R_{s,\theta} \to \sqrt{-2A_s} \) and \( R_{u,\theta} \to R_{u,\infty} = \sqrt{Q_{u,\infty}^{-1} - 2A_u} > 0 \)

3) \( Q_{s,\theta}^{-1} e^{-\theta B_s} \to 0 \), \( P_{s,\theta} = R_{s,\theta}^{-2} Q_{s,\theta}^{-1} e^{-\theta B_s} \to 0 \), \( U_{s,\theta} \to \left( \int_0^{+\infty} e^{\tau B_s} e^{-\tau B_s^*} d\tau \right)^{-1} \)

4) \( Q_{u,\theta} \to Q_{u,\infty} = \int_0^{+\infty} e^{-tB_u} e^{-tB_s^*} dt \) and hence \( P_{u,\theta} = R_{u,\theta}^{-2} Q_{u,\theta}^{-1} e^{-\theta B_u} \to 0 \), \( U_{u,\theta} \to 0 \).

(iii) \( e^{2\theta B_s} \det Q_{s,\theta} \to \det \int_0^{+\infty} e^{\tau B_s} e^{-\tau B_s^*} d\tau \).

**Proof.** For small

\( \theta, Q_{s,\theta} = \theta \left[ 1 - \theta \left( \frac{B_s + B_s^*}{2} \right) + O(\theta^2) \right] \), \( Q_{u,\theta}^{-1} = \frac{1}{\theta} + \frac{B_s + B_s^*}{2} + O(\theta) \).

Similarly \( Q_{u,\theta}^{-1} = \frac{1}{\theta} + \frac{B_u + B_u^*}{2} + O(\theta) \). From these relations it follows that:

\( R_{s,\theta}^2 = \frac{1}{\theta} + \frac{B_s + B_s^*}{2} - 2A_s + O(\theta) \), \( R_{u,\theta}^2 = \frac{1}{\theta} + \frac{B_u + B_u^*}{2} - 2A_u + O(\theta) \),

\( R_{s,\theta}^{-2} = \theta \left[ 1 - \theta \left( \frac{B_s + B_s^*}{2} - 2A_s \right) + O(\theta^2) \right] \), \( R_{u,\theta}^{-2} = \theta \left[ 1 - \theta \left( \frac{B_u + B_u^*}{2} - 2A_u \right) + O(\theta^2) \right] \).

Then:

\( U_{s,\theta} = -2A_s + O(\theta), U_{u,\theta} = -2A_u + O(\theta) \).

Also:

\( R_{s,\theta}^{-2} Q_{s,\theta}^{-1} e^{-\theta B_s} = 1 + O(\theta) \), \( R_{u,\theta}^{-2} Q_{u,\theta}^{-1} e^{-\theta B_u} = 1 + O(\theta) \)

and

\( \det Q_\theta = \theta^{m-1}(1 + O(\theta)) \). 

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As $\theta \to +\infty$, $Q_{s,\theta} = \int_0^\theta e^{-tB_s} e^{-tB_s^*} dt \to +\infty$. Hence $Q_{s,\theta}^{-1} \to 0$, $R_{s,\theta}^2 \to -2A_s$ and $R_{s,\theta} \to \sqrt{-2A_s}$. We have:

$$e^{\theta B_s} Q_{s,\theta} = \left( \int_0^\theta e^{(\theta-t)B_s} e^{(\theta-t)B_s^*} dt \right) e^{-\theta B_s^*}$$

$$Q_{s,\theta}^{-1} e^{-\theta B_s} = e^{\theta B_s^*} \left( \int_0^\theta e^{tB_s} e^{tB_s^*} dt \right)^{-1}.$$ 

It follows that $Q_{s,\theta}^{-1} e^{-\theta B_s} \to 0$ as $\theta \to +\infty$. Also $R_{s,\theta}^{-2} Q_{s,\theta}^{-1} e^{-\theta B_s} \to 0$ and $R_{s,\theta}^{-1} Q_{s,\theta}^{-1} e^{-\theta B_s} \to 0$. It follows that $U_{s,\theta} \to \left( \int_0^{+\infty} e^{tB_s} e^{tB_s^*} dt \right)^{-1}$. As $\theta \to +\infty$, $Q_{u,\theta} \to Q_{u,\infty} = \int_0^{+\infty} e^{-tB_u} e^{-tB_u^*} dt$ and $R_{u,\theta}^2 \to R_{u,\infty}^2 = Q_{u,\infty}^{-1} - 2A_u$. Hence $R_{u,\theta}^{-2} Q_{u,\theta}^{-1} e^{-\theta B_u} \to 0$. Because $U_{u,\theta} = e^{-\theta B_u}(Q_{u,\theta}^{-1} - Q_{u,\theta} R_{u,\theta} Q_{u,\theta}) e^{-\theta B_u}, U_{u,\theta} \to 0$.

Finally

$$e^{\theta B_s} Q_{s,\theta} e^{\theta B_s^*} = \int_0^\theta e^{(\theta-t)B_s} e^{(\theta-t)B_s^*} dt = \int_0^\theta e^{tB_s} e^{tB_s^*} dt$$

and

$$e^{\theta B_s} Q_{s,\theta} e^{\theta B_s^*} \to \int_0^{+\infty} e^{tB_s} e^{tB_s^*} dt$$

$$\det e^{\theta B_s} \det Q_{s,\theta} \det e^{\theta B_s^*} \to \det \int_0^{+\infty} e^{tB_s} e^{tB_s^*} dt.$$ 

But $\det e^{\theta B_s} = \det e^{\theta B_s^*} = e^{\theta \text{Tr} B_s}$. So

$$e^{2\theta \text{Tr} B_s} \det Q_{s,\theta} \to \det \int_0^{+\infty} e^{tB_s} e^{tB_s^*} dt,$$

as $\theta \to +\infty$. This implies that

$$\lim_{\theta \to 0} v(x', \theta) = \tilde{w}(x', 0) \exp -\frac{1}{2} < Ax, x >_{\mathbb{R}^{m-1}} = z(x', 0).$$

**Lemma 10** The solution $z$ does not depend on the unstable variable $x_u$ and is given explicitly by

$$z(x', \theta) = C \exp \left\{ -\frac{1}{4} < \left( \int_0^{+\infty} e^{tB_s} e^{tB_s^*} dt \right)^{-1} x_s, x_s > \right\}.$$
Proof. We have:

\[ v(x', \theta) = \frac{\exp\left[ \theta Tr B_s + \frac{1+O(\theta)}{2} <Ax', x'>_{\mathbb{R}^{m-1}} \right]}{(4\pi \theta)^{\frac{m-1}{2}} (1 + O(\theta))} \int_{\mathbb{R}^{m-1}} \widetilde{w}(\eta + (1 + O(\theta))x', 0) e^{-\frac{1+O(\theta)}{2\theta}(||\eta_s||_{\mathbb{R}^{m}}^2 + ||\eta_u||_{\mathbb{R}^{m}}^2)} d\eta'. \]

Because \( z \) is periodic, for any period \( \theta \), Kolmogorov’s formula (29) gives:

\[ z(x', 0) = \frac{\exp\left[ \theta Tr B_s + \frac{1}{4} <U_{\theta}x', x'>_{\mathbb{R}^{m-1}} \right]}{(4\pi)^{\frac{m-1}{2}}} \sqrt{\det Q_{\theta}} \int_{\mathbb{R}^{m-1}} \widetilde{w}(\eta + R_{\theta}^{-1}P_{\theta}x', 0) e^{-\frac{1}{4}||R_{s}\eta'_{s}||_{\mathbb{R}^{m-1}}^2 - \frac{1}{4}||R_{u}\eta'_{u}||_{\mathbb{R}^{m-1}}^2} d\eta'. \] (33)

Letting \( \theta \to +\infty \), in the expression above and using the estimates of Lemma (9), we see that

\[ z(x', 0) = \frac{\exp\left[ \frac{1}{4} \left( \int_{0}^{+\infty} e^{tB_{s}} e^{tB_{s}^*} dt \right)^{-1} x_s', x'_s >_{\mathbb{R}^{m-1}} \right]}{(4\pi)^{\frac{m-1}{2}}} \sqrt{\det \int_{0}^{+\infty} e^{tB_{s}} e^{tB_{s}^*} dt} \times \int_{\mathbb{R}^{m-1}} \widetilde{w}(\eta', 0) e^{-\frac{1}{4}||R_{s}\eta'_s||_{\mathbb{R}^{m-1}}^2 - \frac{1}{4}||R_{u}\eta'_u||_{\mathbb{R}^{m-1}}^2} d\eta'. \] (34)

Now it is easy to check that the function on the right hand side of (34) is a solution of equation (24). By Tikhonov’s theorem again, this function coincides with \( z \). □

**Corollary 1**  On a cycle \( S \), we have the following decomposition

\[ w_{S}(x', \theta) = z_{S}(x') f_{S}(\theta), \] (35)

where

\[ z_{S}(x') = C e^{-\frac{1}{2} <Ax', x'>} \exp\left\{ -\frac{1}{4} \left( \int_{0}^{+\infty} e^{tB_{s}} e^{tB_{s}^*} dt \right)^{-1} x_s, x'_s > \right\} \] (36)

\[ f_{S}(\theta) = \exp\left\{ -\int_{0}^{\theta} \left[ c_{0}(t) - \left\langle c_{0} \right\rangle_{T_{S}} \right] dt \right\}, \] (37)

where \( C \) is the normalization constant such that \( \sup w_{S} = 1 \).
4.3 Decay estimate of the blown-up function near recurrent sets

To compute the concentration coefficients and to study the convergence properties of the blown-up sequence \( w_\epsilon \), we use the Feynman-Kac formula to compute an asymptotic estimate of \( w_\epsilon \). Then we show that \( w_\epsilon \) converges strongly in \( L^2 \) to its weak limits.

In particular we prove that the sequence \( w_\epsilon \) decays exponentially in the transverse direction of the recurrent set, that can be a critical point, a limit cycle \( S \) or a torus, satisfying the assumptions of paragraph 1.1. The difficulty in obtaining such estimates is caused by the fact that the vector field is not a gradient. In the case of a limit cycle or a torus, the field has two orthogonal components: first a component along the recurrent set (which is zero in the case of a point) and second a transversal one. We shall prove in a coordinate system \( (x', \theta) \) defined in a tubular neighborhood \( T^S \) of \( S \), defined in 1.1, there exist constants \( C > 0 \) and \( C_0 > 0 \), such that

\[
w_\epsilon(x', \theta) = \frac{v_\epsilon(c^{1/2}x', \theta)}{\bar{v}_\epsilon} \leq Ce^{-C_0 \text{dist}_g(x, S)^2} \text{ for } x = (x', \theta) \in T^S.
\]

The proof involves several steps:

- An upper bound of the Fokker-Planck solution (see appendix II). This entails an estimate of the rate function of a Kac-Feynman integral, representing the solution.

- An explicit lower estimate of the rate function \( I_t(x) \) (see definition below) in term of the distance of \( x \) to the limit cycle.

4.3.1 Optimal trajectories of an auxiliary variational problem

Here we present computations of the optimal trajectories associated with the rate function \( I_t(x) \) defined by

\[
I_t(x) = \inf_{\Gamma_{x,t}} \int_0^t \left[ \frac{1}{2} ||\dot{\gamma}(s) + \Omega(\gamma(s))||^2_{g_\gamma(s)} + \Psi_L(\gamma(s)) \right] ds,
\]

where

\[
\Gamma_{x,t} = \{ \gamma \in H^1([0, t]; V) | \gamma(0) = x \}.
\]

Consider the following functional, defined on \( \Gamma_{x,t} \):

\[
I(\gamma) = \int_0^t \left[ \frac{1}{2} ||\dot{\gamma}(s) + \Omega(\gamma(s))||^2_{g_\gamma(s)} + \Psi_L(\gamma(s)) \right] ds.
\]
Then for all \( x \in V, t > 0 \):

\[
I_t(x) = \inf_{I_{t,x}} I(\gamma). \tag{40}
\]

We state the variational problem in the Hamiltonian formalism. The associated Hamiltonian function \( \mathcal{H} : T^*V \to \mathbb{R} \), is:

\[
\mathcal{H}(z) = -<\Omega(\pi_{T^*V}(z)), z > + \frac{1}{2}||z||_{g^*}^2 - \Psi_L(\pi_{T^*V}(z)).
\]

A curve \( z \) is an extremal for the minimization problem (40) if it satisfies the first order condition for an optimum [20]. The Hamiltonian function \( \mathcal{H} : T^*V \to \mathbb{R} \), associated to the problem is:

\[
\mathcal{H}(z) = -<\Omega(\pi_{T^*V}(z)), z > + \frac{1}{2}||z||_{g^*}^2 - \Psi_L(\pi_{T^*V}(z)).
\]

This can also be expressed as

\[
\mathcal{H}(z) = \frac{1}{2}||z - \omega(\pi_{T^*V}(z)||_{g^*}^2 - \frac{1}{2}||\omega(\pi_{T^*V}(z)||_{g^*}^2 - \Psi_L(\pi_{T^*V}(z)),
\]

where \( \omega \) is the 1-form \( G(\Omega) \) associated to the vector field \( \Omega \) (\( G : TV \to T^*V \) is the Legendre transform associated to the metric \( g \)). For any trajectory of the Hamiltonian field \( \overrightarrow{\mathcal{H}} \) of \( \mathcal{H} \), \( z : J \to T^*V, J \) open interval, the function \( t \in J \to \mathcal{H}(z(t)) \in \mathbb{R} \), is constant. Hence for any \( \tau \in J \):

\[
||z(t) - \omega(\pi_{T^*V}(z(t)||_{g^*}^2 = ||\omega(\pi_{T^*V}(z(t)||_{g^*}^2 + 2\Psi_L(\pi_{T^*V}(z(t)) + 2\mathcal{H}(z(\tau)). \tag{41}
\]

Let us call \( F_t \) the flow of \( \overrightarrow{\mathcal{H}} \). In a classical Darboux coordinate system \( z = (x,p), F_t(0,q) = (x^1, \ldots, x^n, p^1, \ldots, p^n) \), these functions satisfy the equations: for \( 1 \leq n \leq m \),

\[
\frac{dx^n}{dt} = \frac{\partial \mathcal{H}}{\partial p_n}(x,p) = -\Omega^n(x) + \sum_{k=1}^{m} g^{kn}(x)p_k, \tag{42}
\]

\[
-\frac{dp_n}{dt} = \frac{\partial \mathcal{H}}{\partial x_n}(x,p) = -\sum_{k=1}^{m} p_k \frac{\partial \Omega^k(x)}{\partial x_n} - \frac{\partial \Psi_L(x)}{\partial x_n} + \frac{1}{2} \sum_{i,j=1}^{m} \frac{\partial g^{ij}(x)}{\partial x_n} p_i p_j, \tag{43}
\]

and \( t \to F_t(0,q) \) is an extremal of the variational problem iff it satisfies the following boundary condition:

\[
p(T; (0,q)) = 0. \tag{44}
\]
Lemma 11  
1. For any trajectory \( z : J \to T^* (V) \) of \( \tilde{\mathcal{H}} \), and any \( t, \tau \in J \),
\[
\| z(t) \|_{g^*}^2 \leq K_0 + 4 \mathcal{H}(z(\tau)) \\
\| \frac{d}{dt} (\gamma) (t) \|_{g}^2 \leq K_1 + 4 \mathcal{H}(z(\tau)),
\]
where
\[
\gamma = \pi_{T^*(V)}(z), \\
K_0 = 4 \sup_V ||\Omega||_g^2 + 2 \sup_V ||\Psi_L||_g^2, \\
K_1 = K_0 + 2 \sup_V ||\Omega||_g^2.
\]

2. For any \( t, \tau \in J \),
\[
\|\tilde{\mathcal{H}}(t)\|_g \leq K_2 + K_3 \mathcal{H}(z(\tau)).
\]
where \( K_2, K_3 \) depend only on \( g, \Omega, \Psi_L \).

Proof.

The first inequality of the lemma is a consequence of expression (41) and
\[
\| z(t) \|_{g^*}^2 \leq 2 \| z(t) - \omega(\gamma(t)) \|_{g^*}^2 + 2 \| \omega(\gamma(t)) \|_{g^*}^2. \tag{45}
\]
The second inequality follows from
\[
\frac{T^\gamma}{dt} = -\Omega(\gamma) + G^{-1} \circ z. \tag{46}
\]
Statement 2 follows from the relation (41)-(42) and statement 1 of the lemma.

Corollary 2  The field \( \tilde{\mathcal{H}} \) is complete.

Proof.

This follows from the fact that the \( \{ \mathcal{H} \leq R^2 \} \) is compact and invariant by the field \( \tilde{\mathcal{H}} \). □

We take a \( q \in V \) and consider a geodesic system of coordinates at \( q, (U, x^1, ..., x^m) \), \( U \) being a geodesic open ball centered at \( q \) and of radius \( \rho \) such that \( \overline{U} \), the closure of \( U \) is contained in \( V - \text{Cut}(x) \). Because \( V \) is compact, we can assume that \( \rho \) is independent of \( q \). Let \( (T^*U, x^1, ..., x^m, p_1, ..., p_m) \) be the associated Darboux chart of \( T^*V \). Let us consider the domain
\[
D_R = \{ z \in T^*(V) | \mathcal{H}(z) \leq R^2 \}.
\]
We have
Lemma 12 1. For $T \geq 0$ such that $T(K_0 + 4R^2) < \rho$, then for any $p \in T^*_x(V)$, for any $t \in [0, T]$,

$$F_t(p) \subset T^*(U).$$

2. There exists a constant $K(R)$ depending only on $g, \Omega, \Psi_L$ and $R$ (but not on $x$) such that in the linear Euclidean structure $T^*(U)$, defined by the coordinate system $(x^1, \ldots, x^n, p_1, \ldots, p_n)$ ($T^*(U) \simeq U \times \mathbb{R}^m \subset \mathbb{R}^{2m}$), we have for any $z_1, z_2 \in D_R$, any $t$ such $t(K_0 + 4R^2) < \rho$

$$||F_t(z_1) - z_1 - (F_t(z_2) - z_2)||_{\mathbb{R}^{2m}} \leq (e^{K(R)t} - 1)||z_1 - z_2||_{\mathbb{R}^m} \quad (47)$$

Proof. The proof is based on Gronwall’s lemma and uses the estimate of lemma 1. 

Let us recall ([20]):

Lemma 13 A curve $z : [0, T] \to T^*(V)$ is an extremal of the optimization problem 40 if

1. $z$ is a trajectory of $\mathcal{H}$ and

2. $z(T) \in 0_{T^*(V)}$.

We now formulate the main proposition of the paragraph

Proposition 4 For any $x \in V$, $T \geq 0$ such that

$$T < \min\{\frac{\rho}{K_0}, \frac{1}{K(0)} \log(3/2)\}, \quad (48)$$

1. There exists a unique curve $\gamma_x \in H^1([0, T]; V)$ such that $\gamma_x(0) = x$ and

$$I_T(x) = I(\gamma_x).$$

2. This curve is located in $U$.

3. $\dot{\gamma}_x(T) + \Omega(\gamma_x(T)) = 0$.

4. For any $u \in H^1([0, T]; T_{\gamma_x}V), u(0) = 0_x \in T_x V, u \neq 0$,

$$\int_0^T \left\{ ||\nabla_{\dot{\gamma}_x(s)}u(s) + \nabla u(s)\Omega||_{g_{\gamma_x(s)}}^2 < \dot{\gamma}_x(s) + \Omega(\gamma_x(s)), \nabla \nabla \Omega(\gamma_x(s))[u(s), u(s)] + g_{\gamma_x(s)} \right\} > \nabla \Psi_L(\gamma_x(s))[u(s), u(s)] ds > 0.$$
Proof.

1) It is easy to see that there exists an constant \( C \) depending only on \( g, \Omega \) and \( \Psi \) such that for all \( T > 0 \) and all \( \gamma \in H^1([0, T]; V) \)

\[
I(\gamma) + CT \geq \frac{1}{4} \int_0^T \left\| \frac{d\gamma}{dt} \right\|_g^2 \, dt.
\]

The existence of a minimizing curve for the functional \( I(\gamma) \) starting at any \( x \in V \) and for any \( T > 0 \) is then standard.

2) If a curve \( \gamma : [0, T] \rightarrow V \) is optimal, then there exists an extremal \( z : [0, T] \rightarrow T^*V \), i.e. a trajectory of the Hamiltonian field \( \overrightarrow{H} \) of \( \mathcal{H} \), lifting \( \gamma \) and such that \( z(T) = 0_{\pi_{T^*V}(z(T))} \) (Lemma 13).

Because \( T \) is small enough (inequality (48)) and \( z(T) = 0_{\pi_{T^*V}(z(T))} \) (Lemma 13),

\[ \mathcal{H}(z(t)) = \mathcal{H}(z(T)) = -\Psi_\xi(\gamma(T)) \leq 0, \]

using Lemma 12, we conclude that \( \gamma([0, T]) \subset U \).

Let \( \gamma_1, \gamma_2 : [0, T] \rightarrow V \) be two extremal curves starting at \( x \). Let \( z_1, z_2 : [0, T] \rightarrow T^*(V) \) be their liftings. By the preceding considerations \( z_i([0, T]) \subset U \), for \( i=1,2 \). Choose a \( R > 0 \), such that \( TK(R) < \log(3/2) \), \( T(K_0 + 4R^2) < \log(3/2) \), \( z_i(0) \subset D_0 \subset D_R \). By relation (47),

\[ \left\| (F_t(z_1(0)) - z_1(0)) - (F_t(z_2(0)) - z_2(0)) \right\|_{\mathbb{R}^{2m}} \leq \left( e^{K(R)t} - 1 \right) \left\| z_1(0) - z_2(0) \right\|_{\mathbb{R}^m} (49) \]

Let us call \( \Pi_2 : T^*(U) \rightarrow T^*V \) the canonical projection related to the product structure \( T^*(U) \simeq U \times \mathbb{R}^m \).

\[
\left\| \Pi_2((F_t(z_1(0)) - z_1(0)) - (F_t(z_2(0)) - z_2(0))) \right\|_{\mathbb{R}^{2m}} \leq \left\| \Pi_2(F_t(z_1(0)) - z_1(0)) - (F_t(z_2(0)) - z_2(0)) \right\|_{\mathbb{R}^{2m}},
\]

\[
\left\| \Pi_2((F_t(z_1(0))) - z_1(0)) - \Pi_2((F_t(z_2(0))) - z_2(0))) \right\|_{\mathbb{R}^{2m}} \leq \frac{1}{2} \left\| (z_2(0)) - z_1(0) \right\|_{\mathbb{R}^{2m}}.
\]

Thus

\[
\left\| \Pi_2((F_t(z_1(0))) - \Pi_2((F_t(z_2(0))) \right\|_{\mathbb{R}^{2m}} \geq \frac{1}{2} \left\| (z_2(0)) - z_1(0) \right\|_{\mathbb{R}^{2m}}.
\]

By paragraph 2), \( F_t(z_i(0)) = z_i(T) \in 0_{\pi_{T^*V}(z(T))} \), which implies that \( \Pi_2((F_t(z_1(0))) = 0_x \) and thus \( z_2(0) = z_1(0) \).

The proofs of part 2) 3) are consequences of Lemma 12 and Lemma 13 respectively. We now prove Part (4) of proposition 4: There exists a constant \( K_3 \)
depending only on \( g, \Omega \) and \( \Psi \) such that for all \( s \in [0, t] \), all \( u \in T_{g, \Omega}^* V \)

\[
< \dot{\gamma}_x(s) + \Omega(\gamma_x(s)), \nabla \nabla \Omega(\gamma_x(s))[u, u] >_{g_x(s)} + \nabla d\Psi(\gamma_x(s))[u, u] \leq K_3 \| u \|^2_{g_x(s)}
\] (50)

Also for any \( u \in H^1([0, t]; T_{g_x} V) \) such that \( u(0) = 0_x \)

\[
\int_0^T \| u(s) \|^2_{g_x(s)} ds \leq T^2 \int_0^T \| \nabla g_x(s) u(s) \|^2_{g_x(s)} ds \quad (51)
\]

Suppose that there exists a \( \bar{\pi} \in H^1([0, T]; T_{g_x} V), \bar{\pi}(0) = 0_x \in T_x V \), \( \bar{\pi} \neq 0 \) and

\[
\int_0^T \left\{ \| \nabla \gamma_x(s) \bar{\pi}(s) + \nabla \bar{\pi}(s) \Omega \|^2_{g_x(s)} + < \dot{\gamma}_x(s) + \Omega(\gamma_x(s)), \nabla \nabla \Omega(\gamma_x(s))[\bar{\pi}(s), \bar{\pi}(s)] >_{g_x(s)} + \nabla d\Psi(\gamma_x(s))[\bar{\pi}(s), \bar{\pi}(s)] ds \right\} \leq 0
\] (52)

Then by (50),(51) and (52)

\[
\int_0^T \| \nabla \gamma_x(s) \bar{\pi}(s) + \nabla \bar{\pi}(s) \Omega \|^2_{g_x(s)} ds \leq K_3 \int_0^T \| \bar{\pi} \|^2_{g_x(s)} ds.
\] (53)

Moreover,

\[
\int_0^T \| \nabla \gamma(s) \bar{\pi}(s) \|^2_{g_x(s)} ds \leq 2 \int_0^T \| \nabla \gamma_x(s) \bar{\pi}(s) + \nabla \bar{\pi}(s) \Omega \|^2_{g_x(s)} ds + 2 \int_0^T \| \nabla \bar{\pi}(s) \Omega \|^2_{g_x(s)} ds.
\]

There is a constant \( K_4 \) depending only on \( g, \Omega \) such that

\[
\int_0^T \| \nabla \bar{\pi}(s) \Omega \|^2_{g_x(s)} ds \leq K_4 \int_0^T \| \bar{\pi} \|^2_{g_x(s)} ds.
\] (54)

Thus from (51),(53) and (54) we get

\[
\int_0^T \| \nabla \gamma(s) \bar{\pi}(s) \|^2_{g_x(s)} ds \leq 2(K_3 + K_4) \int_0^T \| \bar{\pi} \|^2_{g_x(s)} ds \leq 2(K_3 + K_4) T^2 \int_0^T \| \nabla \gamma(s) \bar{\pi}(s) \|^2_{g_x(s)} ds.
\]

For \( T \sqrt{2(K_3 + K_4)} < 1 \), we get a contradiction. This ends the proof of proposition 4.  

4.3.2 Explicit decay estimate of the eigenfunction

Let \( \Theta_\varepsilon : \mathbb{R}_+ \times V \rightarrow \mathbb{R} \), is the function \( e^{-\lambda t} v_\varepsilon(x) \), then we have

\[
\varepsilon \Delta_\varepsilon \Theta_\varepsilon + \theta(\varepsilon) \Theta_\varepsilon + \frac{c_\varepsilon}{\varepsilon} \Theta_\varepsilon + \frac{\partial \Theta_\varepsilon}{\partial t} = 0, \text{ on } \mathbb{R}_+^* \times V \quad (55)
\]

\[
\Theta_\varepsilon(0, x) = v_\varepsilon(x) \text{ for } x \in V.
\]

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Recall the Feynman-Kac formula:

$$\Theta_{\epsilon}(t,x) = \mathbb{E}_x \left[ v_{\epsilon}(X_{x}^{\epsilon}(t)) \exp - \int_0^t \frac{c_{\epsilon}(X_{x}^{\epsilon}(s))}{\epsilon} ds \right],$$

$$\frac{\Theta_{\epsilon}(t,x)}{\nu_{\epsilon}} = \mathbb{E}_x \left[ \frac{v_{\epsilon}(X_{x}^{\epsilon}(t))}{\nu_{\epsilon}} \exp \left( - \int_0^t \left\{ c(X_{x}^{\epsilon}(s)) + \frac{\Delta_g \mathcal{L}(X_{x}^{\epsilon}(s))}{2} \right\} ds \right) \exp - \int_0^t \frac{\Psi_{\epsilon}(X_{x}^{\epsilon}(s))}{\epsilon} ds \right].$$

(56)

$X_{x}^{\epsilon}$ is the diffusion process on $V$ having $-\epsilon \Delta g - \theta(\Omega)$ as generator.

To estimate the right hand side of the relation (56) when $\epsilon \to 0$, we are going to follow the method presented in the papers ([2, 4]). For $0 < t < T$, the function $I$ has a unique minimum point on the set

$$C^0([0,t]; V, x) = \{ f : [0, t] \to V | f \text{ continuous, } f(0) = x \},$$

(57)

namely $\gamma_x$, which is non degenerate by Proposition 4 (part-3). We are now ready to announce and prove the main result of this section:

**Theorem 5** For $0 < t < T$ there exists a constant $C_t$ depending only on $g, \Omega$ and $\Psi$ such that for

$$\frac{v_{\epsilon}(x)}{\nu_{\epsilon}} \leq C_t \exp \left[ t\lambda_{\epsilon} - \frac{I_t(x)}{2\epsilon} \right].$$

(58)

**Proof:** Recall that $X_{x}(t)$ is a stochastic process on $V$ having $-\epsilon \Delta g - \theta(b)$ as generator, $X_{x}^{\epsilon}(t)$ the process starting at $x \in V$. Choose $x \in V$.

$$\frac{v_{\epsilon}(x)}{\nu_{\epsilon}} = \mathbb{E}_x \left[ \frac{v_{\epsilon}(X_{x}^{\epsilon}(t))}{\nu_{\epsilon}} \exp \left( - \int_0^t \left\{ c(X_{x}^{\epsilon}(s)) + \frac{\Delta_g \mathcal{L}(X_{x}^{\epsilon}(s))}{2} \right\} ds \right) \exp - \int_0^t \frac{\Psi_{\epsilon}(X_{x}^{\epsilon}(s))}{\epsilon} ds \right].$$

(56)

Let us define

$$\tilde{E}_h = \{ \gamma \in H^1([0,t]; V, x) | \int_0^t \frac{1}{2} ||\dot{\gamma}(s) + \Omega(\gamma(s))||_{g_{\gamma(s)}} ds \leq h \}. \quad (59)$$

We call $\chi_1, \chi_2, \chi_3$ the characteristic functions of the subsets of $C^0([0,t]; V, x)$,

$$\{ \gamma | d_{\infty}(\gamma; \tilde{E}_h) > \delta, d_{\infty}(\gamma; \gamma_x) \geq \eta \}, \quad (60)$$

$$\{ \gamma | d_{\infty}(\gamma; \tilde{E}_h) \leq \delta, d_{\infty}(\gamma; \gamma_x) \geq \eta \}, \quad (61)$$

$$S = \{ \gamma | d_{\infty}(\gamma, \gamma_x) < \eta \} \quad (62)$$
respectively and the distance is

\[ d_\infty(\gamma_1, \gamma_2) = \sup_{[0,t]} d_g(\gamma_1(s), \gamma_2(s)). \] (63)

Set \( F(X_\varepsilon) = \frac{v_\varepsilon(X_\varepsilon^2(t))}{v_\varepsilon} \exp \left( -\int_0^t \left\{ c(X_\varepsilon^2(s)) + \frac{\Delta_x L(X_\varepsilon^2(s))}{2} \right\} ds \right) \exp -\int_0^t \frac{\psi_\varepsilon(X_\varepsilon^2(s))}{\varepsilon} ds. \)

Then:

\[
\frac{v_\varepsilon(X_\varepsilon^2(t))}{v_\varepsilon} = E_x \left[ F(X_\varepsilon)\chi_1(X_\varepsilon) \right] + E_x \left[ F(X_\varepsilon)\chi_2(X_\varepsilon) \right] + E_x \left[ F(X_\varepsilon)\chi_3(X_\varepsilon) \right]. \tag{64}
\]

Recall that

\[
N : \gamma \in C^0([0, t]; V) \rightarrow \left\{ \begin{array}{ll}
\int_0^t \frac{1}{2} \left\| \dot{\gamma}(s) + \Omega(\gamma(s)) \right\|^2_{g_\varepsilon(s)} ds & \text{if } \gamma \in H^1([0, t]; V) \\
+\infty & \text{if } \gamma \in H^1([0, t]; V)
\end{array} \right.
\]

is the rate function for \( X_\varepsilon \). Then it follows from the theory of large deviations that for some constants \( M_1, l_1 > 0 \) and \( \varepsilon_1 > 0 \): for \( 0 < \varepsilon \leq \varepsilon_1 \),

\[
|E_x \left[ F(X_\varepsilon)\chi_1(X_\varepsilon) \right]| \leq M_1 P_x(X_\varepsilon) \in \{ \gamma | d_\infty(\gamma, E_h) \geq \delta \} \leq M_1 \exp - \frac{l_1(x)}{2\varepsilon t}.
\]

The subset \( \{ \chi_2 = 1 \} \) of \( C^0([0, t]; V, x) \) is compact for the weak topology. On the other hand \( I \) is lower semi-continuous for the same topology. Hence it attains its minimum on the compact \( \{ \chi_2 = 1 \} \). Because \( \gamma_x \) is the only minimum point of \( I \) and \( \{ \chi_2 = 1 \} \subset \{ \gamma | d_\infty(\gamma, \gamma_x) \geq \eta \} \), \( \min_{\chi_2 = 1} I \geq l_1(x) + l_2 \) for some constant \( l_2 \). Then there exists a constant \( M_2 > 0 \) such that

\[
E_x \left[ F(X_\varepsilon)\chi_2(X_\varepsilon) \right] \leq M_2 \exp - \frac{l_1(x) + l_2}{2\varepsilon t}.
\]

For these two estimates see [3]. The study of the last term on the right hand side of the relation (64) is more involved. By construction \( \gamma_x \) is contained in a geodesic coordinate chart \( (U, \xi^1, \ldots, \xi^m) \) with pole at \( x \). Choosing \( \delta \) sufficiently small we can assume that \( \{ \gamma | d_\infty(\gamma, \gamma_x) < \eta \} \subset U \). Then we can take advantage of the linear structure induced on \( U \) by the coordinate system. We can consider the process \( X_\varepsilon \) on \( U \) defined in the coordinate system \( (U, \xi^1, \ldots, \xi^m) \) by the stochastic equation:

\[
dX_\varepsilon(t) = -\Omega_\varepsilon(X_\varepsilon(t)) + \sqrt{\varepsilon}\sigma(X_\varepsilon(t))dw(t)
\]

where \( \Omega_\varepsilon(x) = \Omega(x) + \varepsilon \sum_{i,j} g^{ij}(x) \Gamma^k_{ij}(x)e_k \) and \( \sigma : U \rightarrow GL(m; \mathbb{R}) \) is a \( C^\infty \) function such that \( \sigma\sigma^*= g \) and \( w \) is a standard brownian motion. Note that there exists a constant \( C > 0 \) such that

\[
E_x \left[ F(X_\varepsilon)\chi_3(X_\varepsilon) \right] \leq CE_x \left\{ \chi_3(X_\varepsilon)e^{-\frac{1}{\varepsilon}\Psi_\varepsilon(X_\varepsilon)} \right\}.
\]

For the estimate of \( E_x \left\{ \chi_3(X_\varepsilon)e^{-\frac{1}{\varepsilon}\Psi_\varepsilon(X_\varepsilon)} \right\} \), we refer to the appendix II.
Lemma 14 For fixed $t \in [0, \overline{T}]$, there exists a constant $C_t > 0$ such that for all $x \in T^S$ (the tubular neighborhood of $S$ defined in paragraph 1.1),

$$I_t(x) \geq C_t \|x'\|^2_{\mathbb{R}^{m-1}}.$$  

Proof. To prove this lemma we restate the variational problem as a optimal control problem and study the subspace $E_t$ of $C^0([0, t]; T^*V)$ consisting of the extremal trajectories $z : [0, t] \to T^*V$ of the problem. Let $T^S$ be a tubular neighborhood of the limit cycle $S$ endowed with a cyclic coordinate system $(x', \theta)$ as in paragraph 1.1. Recall that we use the notations $x' = (x^1, ..., x^{m-1}), \theta = x^m$ as in paragraph 1.1. Using this coordinate systems, we shall identify the tangent and cotangent spaces $TT^S$ and $T^*T^S$ with $T^S \times \mathbb{R}^m$. An element $z \in T^*T^S$ will be represented by a couple $(x, p)$ and an element $\tau \in TT^S$ by a couple $(x, u)$. It is easy to see that $E_t$ is relatively compact in $C^0([0, t], T^*V)$ (Because the final point is in the zero section which is compact and the derivative of the extremals are bounded along $[0, T]$). Let $z \in E_t$.

Set $\pi_T \circ z = \gamma$. Define

$$\hat{I}(z) = \int_0^t \left[ \frac{1}{4} \|z(s)\|_{g^*}^2 + \Psi_L(\gamma(s)) \right] ds$$  

(65)

The functional $\hat{I} : E_t \to \mathbb{R}$ is clearly continuous. Let $E_t^S$ be the subset of $E_t$ of those $z$ such that $z(0) \in T^*T^S$.

Define the functional $In : E_t^S \to \mathbb{R}$, by setting $In(z) = \|\gamma'(0)\|^2_{\mathbb{R}^{m-1}}$ where $\gamma'(0) = (x^1(z(0)), \ldots, x^{m-1}(z(0)))$. The quotient $In/\hat{I}$ is continuous on $E_t^S - In^{-1}(0)$ for the $C^0$ topology.

If this quotient is not bounded on $E_t^S - In^{-1}(0)$, then there exists a sequence $\{z^k | k \in \mathbb{N}\}$ in $E_t^S - In^{-1}(0)$ such for each $k \in \mathbb{N}$

$$\frac{In(z^k)}{\hat{I}(z^k)} \geq k.$$  

(66)

Recall that $\Psi_L \geq 0$. It follows from relations (65) and (66) that for all $k \in \mathbb{N}$.

$$\int_0^t \frac{1}{4} \|z^k(s)\|_{g^*}^2 ds \leq \frac{1}{k} In(z^k), \quad \int_0^t \Psi_L(\gamma^k(s)) ds \leq \frac{1}{k} In(z^k),$$  

(67)

and

$$|\mathcal{H}(z^k(t))| = \Psi_L(\gamma^k(t)).$$  

(68)

For all $s \in [0, t]$

$$\|\gamma^k(0) - \gamma^k(s)\|^2_{\mathbb{R}^{m-1}} \leq s \int_0^s \left\| \frac{d\gamma^k(\sigma)}{d\sigma} \right\|^2_{\mathbb{R}^{m-1}} d\sigma.$$  

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Using the relation (46), noting that \( \Omega^q(0, \theta) = 0 \) if \( 1 \leq q \leq m - 1 \), for an appropriate constant \( C_2 \) depending only on \( g \) and \( \Omega \)

\[
\|\gamma^k(0) - \gamma^k(s)\|_{R^{m-1}}^2 \leq sC_2 \left( \int_0^s \frac{1}{4} \|z^k(\sigma)\|_{\gamma^*}^2 d\sigma + \int_0^s \|\gamma^k(\sigma)\|_{R^{m-1}}^2 d\sigma \right)
\]

for all \( s \in [0, t] \). By the assumptions on \( \Psi_L \) there exists a constant \( C_3 > 0 \) depending only on \( \Psi \) such that for all \( s \in [0, t] \), all \( k \in \mathbb{N} \)

\[
\frac{\|\gamma^k(s)\|_{R^{m-1}}^2}{C_3} \leq \Psi_L(\gamma(s)) \leq C_3 \|\gamma^k(s)\|_{R^{m-1}}^2.
\]

Hence for \( C_4 = \max(C_2, C_2C_3) \), for all \( s \in [0, t] \)

\[
\|\gamma^k(0) - \gamma^k(s)\|_{R^{m-1}}^2 \leq sC_4 \left( \int_0^s \frac{1}{4} \|z^k(\sigma)\|_{\gamma^*}^2 d\sigma + \int_0^s \Psi_L(\gamma^k(\sigma)) d\sigma \right)
\]

Using equations (67), for all \( s \in [0, t] \)

\[
\|\gamma^k(0) - \gamma^k(s)\|_{R^{m-1}}^2 \leq sC_4 \frac{1}{k} In(z^k),
\]

\[
\|\gamma^k(0)\|_{R^{m-1}}^2 \leq 2\|\gamma^k(s)\|_{R^{m-1}}^2 + 2sC_4 \frac{1}{k} In(z^k).
\]

Integrating on \([0, t]\)

\[
t\|\gamma^k(0)\|_{R^{m-1}}^2 \leq 2 \int_0^t \|\gamma^k(s)\|_{R^{m-1}}^2 ds + t^2C_4 \frac{1}{k} In(z^k).
\]

Using the relations (69),(67) for all \( k \in \mathbb{N} \)

\[
\int_0^t \|\gamma^k(s)\|_{R^{m-1}}^2 ds \leq C_3 \int_0^t \Psi_L(\gamma(s)) ds \leq C_3 \frac{1}{k} In(z^k),
\]

\[
t\|\gamma^k(0)\|_{R^{m-1}}^2 \leq (2C_3 + t^2C_4) \frac{1}{k} In(z^k),
\]

\[
\|\gamma^k(0)\|_{R^{m-1}}^2 \leq (\frac{2C_3}{t} + tC_4) \frac{1}{k} In(z^k),
\]

\[
In(z^k) \leq (\frac{2C_3}{t} + tC_4) \frac{1}{k} In(z^k).
\]

For \( k \) so large that \((\frac{2C_3}{t} + tC_4) \frac{1}{k} < 1 \) we get a contradiction. This ends the
proof of the lemma. ■

Using the result of the lemma, we conclude that
\[
\frac{v_\varepsilon(x)}{v_\varepsilon} \leq C_t \exp \left[ t\lambda_\varepsilon - \frac{I_1(x)}{2\varepsilon} \right] \\
w_\varepsilon(x) \leq C_1 \exp \left[ -C_2 \|x^\prime\|^2_{R^{m-1}} \right].
\] (70)

\(C_1, C_2\) are two positive constants independent of \(\varepsilon\). In particular, we have for a point, a cycle or a torus belonging to the recurrent set,
\[
w_\varepsilon(x) \leq C_1 \exp \left[ -C_2 \text{dist}_g(x, S)^2 \right].
\] (71)

**Corollary 3** Any sequence of \(\varepsilon\)’s converging to zero contains a subsequence \(\varepsilon_n\) such that \(w_\varepsilon\) converges in \(L_2\) to the blown up limit function \(w\).

The proof follows from lemma (6) and inequality (70).

### 4.4 Absolute continuity of the limit measures on the limit cycle

In this paragraph, as stated at the beginning of section 4, we shall prove that along a limit cycle, the limit measures are absolutely continuous with respect to the length. The proofs are based on the decay estimates, established in the previous paragraph 4.3.

When a cycle \(S\) is charged (see definition 3), we shall prove that:

**Proposition 5** The restriction of a limit measure \(\mu\) to a cycle \(S\) is absolutely continuous with respect to the length.

**Proof.** Let \(\{v_\varepsilon|n \in \mathbb{N}\}, \varepsilon_n \longrightarrow 0\) as \(n \longrightarrow \infty\), be a sequence such that the sequence of measures \(\left\{\frac{v^2_\varepsilon dV_g}{\int v^2_\varepsilon dV_g} | n \in \mathbb{N}\right\}\) converges weakly to \(\mu\). For simplicity we shall assume that the \(v_\varepsilon\) are normalized (\(\int v^2_{\varepsilon_n} dV_g = 1\)). Let \((U, x_1, \ldots x_{m-1}, \theta)\) be an adapted coordinate system for the cycle \(S\) as in paragraph 1.1 such that \(U \supset T(\delta)\) the image of which is the set \(\{\sum_{j=1}^{m-1} x_j^2 \leq \delta^2\}\). Let \(\varphi\) be a continuous function defined in the tubular neighborhood \(T(\delta)\). Then:

\[
\int_S \varphi d\mu = \lim_{n \to +\infty} \int_{T(\delta)} \varphi v^2_{\varepsilon_n} dV_g.
\]

Take \(\varphi(x)\) of the form \(\psi(x^\prime)\eta(\theta)\), \(x^\prime = (x_1, \ldots, x_{m-1})\). Using the blow-up analysis of theorem 2, for all \(\delta > 0\), sufficiently small

\[
\int_{T(\delta)} \varphi v^2_{\varepsilon_n} dV_g = v^2_{\varepsilon_n} \varepsilon_n^{m-1} \int_0^{T_S} \int_{B^{m-1}(\delta/\sqrt{\varepsilon_n}) \times \{\theta\}} \psi(x^\prime \sqrt{\varepsilon_n}) \eta(\theta) w^2_\varepsilon(x^\prime, \theta) d\Sigma_g d\theta,
\]
where $d\Sigma_{g_{\bar{\varepsilon}n}}$ is the measure induced on each n-1 dimensional ball $B^{m-1}(\delta)$ of radius $\delta$, transverse to the cycle by the blown-up metric $g_{\bar{\varepsilon}n}$.

Since the sequence $\bar{v}_{\varepsilon_{n}}^{2} \varepsilon_{n}^{m-1}$ converges as $\varepsilon_{n}$ goes to zero (see the next subsection), the blow-up analysis of theorem 2 and the decay estimate along the cycle show that the integral

$$\int_{0}^{l} \int_{B^{m-1}(\delta/\sqrt{\varepsilon_{n}}) \times \{\theta\}} \psi(x' \sqrt{\varepsilon_{n}})\eta(\theta) w_{\varepsilon_{n}}^{2}(x', \theta) d\Sigma_{g_{\bar{\varepsilon}n}} d\theta,$$

converges as $\varepsilon_{n}$ goes to zero. Indeed, $w_{\varepsilon_{n}}$ converges strongly in $L^{2}(\mathbb{R}^{m-1} \times S^{1})$ to a blown-up function $w_{S}$ and by Fubini theorem,

$$\lim_{n \rightarrow +\infty} \int_{B^{m-1}(\delta/\sqrt{\varepsilon_{n}})} \varphi w_{\varepsilon_{n}}^{2} d\Sigma_{\epsilon} = \int_{0}^{T_{S}} \int_{\mathbb{R}^{m-1}} w_{S}^{2}(x', \theta) \psi(0)\eta(\theta) dx' d\theta = \int_{\mathbb{R}^{m-1}} \psi(0)\eta(\theta) \int_{0}^{l} w_{S}^{2}(x', \theta) dx' d\theta.$$

If $\delta_{x'}$ denotes the Dirac measure in the transverse variable $x'$ only, the normalized eigenfunction sequence converges in the distribution sense and on the cycle $S$,

$$\lim_{n \rightarrow +\infty} \int_{\mathbb{R}^{m-1}} \psi(0)\eta(\theta) w_{S}^{2}(x', \theta) dx' d\theta = \int_{\mathbb{R}^{m-1}} \psi(0) w_{S}^{2}(x') dx' \int_{0}^{T_{S}} f_{S}^{2}(\theta)\eta(\theta) d\theta.$$

Another representation of the continuous component of the limit measure is

$$\frac{d\mu}{d\theta} = f_{S}^{2}(\theta) = \lim_{n \rightarrow +\infty} \int_{H_{l}} v_{\varepsilon}^{2} d\Sigma_{g}$$

where $H_{l}$ denotes any hypersurface cutting the orbit at the point of abscissa $\theta = l$ transversally. The limit is independent of the choice of the hypersurface.

4.5 The limits of $\sup_{V_{m}} v_{\varepsilon}$

Let us recall that $\bar{v}_{\varepsilon_{n}} = \sup_{V_{m}} v_{\varepsilon_{n}}$. We shall now prove that either $\bar{v}_{\varepsilon_{n}}^{2} \varepsilon_{n}^{m-1}$ converges to zero and then no critical point is charged. In that case $\bar{v}_{\varepsilon_{n}}^{2} \varepsilon_{n}^{m-1}$ converges to a strictly positive constant and at least one cycle is charged.
To prove this statement, we use the $L_2$ normalization condition

$$1 = \int_{V_m} v_{\varepsilon_n}^2 dV_g$$

and we perform the blow-up change of coordinates in the neighborhood of the recurrent set. Outside any neighborhood of the recurrent set, the sequence $v_{\varepsilon_n}$ converges to zero in $L_2$ (see [24]).

$$1 = \int_{V_m} v_{\varepsilon_n}^2 = \overline{v}_{\varepsilon_n}^2 \frac{m-1}{n} \left( \sum_S \int_{T_S(\delta/\varepsilon_n^{1/2})} w_{\varepsilon_n}^2(x) dV_g + \varepsilon_n^{1/2} \sum_P \int_{B_P(\delta/\varepsilon_n^{1/2})} w_{\varepsilon_n}^2(x) dV_g \right) + \int_{V'_m} v_{\varepsilon_n}^2$$

where

$$V'_m = V_m - \bigcup_S T_S(\delta/\varepsilon_n^{1/2}) - \bigcup_P B_P(\delta/\varepsilon_n^{1/2}).$$

The first sum on the right hand side is extended over the limit cycles and the second over the stationary points.

The last integral in the r.h.s. of equation (73) tends to zero, because we integrate over an open set whose closure does not intersect the recurrent set (see the proof in [24]).

After selecting a subsequence if necessary, there exists a component $C$ of the recurrent set and a sequence $Q_n$ such that

1. $v_{\varepsilon_n}(Q_n) = \overline{v}_{\varepsilon_n}$.
2. $Q_n \rightarrow Q_\infty \in C$.
3. $\frac{Q_n - Q_\infty}{\sqrt{\varepsilon_n}} \rightarrow Q_\ast$. This difference is taken in the linear structure defined by the coordinate system introduced in 1.1. $Q_\ast$ belongs to the blow-up space $\mathbb{R}^m$ or $\mathbb{R}^m \times S^1$.

Let us show that $w$ is not identically 0. We know that it converges uniformly on any compact set $a$ to $w_C$ and

$$\lim_{n \rightarrow \infty} w_{\varepsilon_n} \left( \frac{Q_n - Q_\infty}{\sqrt{\varepsilon_n}} \right) = w_C(Q_\ast) = 1.$$  

We conclude that $w_C$ is not identically 0. If $C$ is a point, using Corollary 3 we have that $w_{\varepsilon_n}$

$$\lim_{n \rightarrow +\infty} \int_{B_C(\delta/\varepsilon_n^{1/2}} w_{\varepsilon_n}^2(x) dV_g = \int_{\mathbb{R}^m} w_C^2(x) dx > 0.$$  

If $C$ is a cycle, we have

$$\lim_{n \rightarrow +\infty} \int_{T_C(\delta/\varepsilon_n^{1/2}} w_{\varepsilon_n}^2(x) dV_g = \int_{\mathbb{R}^{m-1} \times S^1} w_C^2(x', \theta) dx' d\theta > 0.$$  

We conclude using formula (73) and the previous identities (76) or (77), that after selecting a subsequence if needed.

If $C$ is a point:

$$\lim_{n \to +\infty} \bar{v}_n^2 \varepsilon_{n}^2 > 0. \quad (78)$$

If $C$ is cycle,

$$\lim_{n \to +\infty} \bar{v}_n^2 \varepsilon_{n}^m \varepsilon_{n}^{-1} > 0. \quad (79)$$

When a cycle is charged, it follows from formula (73) that no point is charged.

For any component $S$ of the recurrent set, selecting a subsequence if needed:

If $S$ is a cycle,

$$\int_{T_S(\delta/\varepsilon_n^{1/2})} w_{\varepsilon_n}^2(x) dV_{ge} = C^{-1}(b, g) \gamma_S^2 \int_{\mathbb{R}^{m-1} \times S^1} w_C^2(x, \theta) dx d\theta, \quad (80)$$

$$\lim_{\varepsilon \to 0} \varepsilon^{(m-1)/2} \bar{v}_\varepsilon^2 = C(b, g).$$

If $S$ is a point, we have

$$\int_{B_S(\delta/\varepsilon_n^{1/2})} w_{\varepsilon_n}^2(x) dV_{ge} = C^{-1}(b, g) \gamma_S^2 \int_{\mathbb{R}^m} w_C^2(x) dx, \quad (81)$$

$$\lim_{\varepsilon \to 0} \varepsilon^{(m-1)/2} \bar{v}_\varepsilon^2 = C(b, g).$$

If no cycle is charged, then

$$C(b, g) = \sum_{S \in \mathcal{C}} \gamma_S^2 \int_{\mathbb{R}^m} w_C^2(x) dx,$$

If at least one cycle is charged then

$$C(b, g) = \sum_{S \in \mathcal{C}} \gamma_S^2 \int_{\mathbb{R}^{m-1} \times S^1} w_C^2(x, \theta) dx d\theta.$$

When several cycles are charged, using expression (73), we obtain an exact expression of the limit measure $\mu$. If $\mathcal{C}$ denotes the set of charged limit cycles then

$$\mu = \frac{\sum_{S \in \mathcal{C}} \gamma_S^2 \left( \int_{\mathbb{R}^{m-1}} z_S^2(x') dx' \right) \delta_{x'} \otimes f_S^2(\theta) d\theta}{\sum_{S \in \mathcal{C}} \gamma_S^2 \left( \int_{\mathbb{R}^{m-1}} z_S^2(x') dx' \right) f_T^0 f_S^2(\theta) d\theta},$$

where we have used the result of corollary 1. When no cycles are charged, we have

$$\mu = \frac{\sum_{P \in \mathcal{C}_{sing}} \gamma_P^2 \left( \int_{\mathbb{R}^m} z_P^2(x) dx \right) \delta_P}{\sum_{P \in \mathcal{C}_{sing}} \gamma_P^2 \left( \int_{\mathbb{R}^m} z_P^2(x) dx \right)},$$

where $\mathcal{C}_{sing}$ denotes the set of charged critical points where the topological pressure is achieved. This ends the proof of theorem 4 parts 2-3-4 of theorem 3. ■
5 Concentration on two-dimensional torus

In this section, we shall study the limit measures of the sequence $v_\epsilon^2 dV_g$, when the recurrent set of the vector field $b$ contains hyperbolic two dimensional torii. We shall not try to extend our results, when the recurrent sets are of dimension $n \geq 3$, but it is likely that similar results are valid on $n$ dimensional manifolds under some restrictive assumptions. The characterization of the limit measures remains an open problem, even in dimension 3.

However, we will examine the concentration of eigenfunctions along two dimensional torii and show that the limit measures are absolutely continuous with respect to the probability measure on the torus invariant by the flow of $b$ (see assumption IV in 1.1: the restriction of the field to the torus generates an irrational flow).

5.1 Main result

Theorem 6 On a Riemannian manifold $(V, g)$, let $b$ be a hyperbolic vector field such that the recurrent set $\mathcal{R}$ consists of points $P_1, ..., P_M$, cycles $S_1, ..., S_N$ and two-dimensional irrational torii $T_1, ..., T_r$ and assumptions I-IV of section 1.1 are satisfied.

Let $u_\epsilon > 0$ be the first eigenfunction of $L_\epsilon$, normalized using the $L_2$-norm. We denote by $L$, a special Lyapunov function. The set of weak limits of the probability measures $\frac{u_\epsilon^2 e^{-L/\epsilon}}{\int_V u_\epsilon^2 e^{-L/\epsilon} dV_g}$ when $\epsilon$ goes to zero is contained inside the set

$$\Lambda_2 = \{ \mu = \sum_{P \in S_{tp}} c_P \delta_P + \sum_{T \in S_{tp}} b_T \delta_T, c_P \geq 0, a_T \geq 0, b_T \geq 0 \},$$

where

$$\delta_T(h) = \int_T h(\theta_1, \theta_2) f_T(\theta_1, \theta_2) dS_T,$$

$dS_T = \frac{d\theta_1 d\theta_2}{\int d\theta_1 d\theta_2}$ is the two-dimensional normalized measure on $T$ invariant under the action of the field $b$ and $f_T$ is the unique solution of maximum 1, of the equation

$$k_1 \frac{\partial f}{\partial \theta_1} + k_2 \frac{\partial f}{\partial \theta_2} + cf = \mu_2 f \text{ on } T,$$

$$\mu_2 = \int_T c dS_T.$$

The measures $\delta_P$, $\delta_T$ were defined in the previous sections.

When the topological pressure is achieved at least on one torus, the maximum of the sequence $v_\epsilon = u_\epsilon e^{-L/2\epsilon}$ goes to infinity and

$$\lim_{\epsilon \to 0} e^{n/2-1} v_\epsilon^2 = C(\Omega, c),$$
where the constant $C(\Omega, c)$ is given by

$$C(\Omega, c) = \frac{1}{\sum_{T \in S'} \gamma_T^2 \int_T f_T dS_T},$$

the sum is extended to all torus where topological pressure is achieved and $\gamma_T$ are the modulating coefficients. If the topological pressure is not attained on any torus, then the torii do not contribute to the limit measures.

**Remark 1.** If at least one torus is charged, the critical points and limit cycles do not contribute to the limit measures. On the other hand, if no torii are charged, the description of the limit measures is the same as when no torii are present.

**Remark 2.** Instead of two dimensional torii, we could have considered more general compact surfaces in $V$ invariant by the flow of $b$. But we do not know what assumptions should be made on the flow restricted to the surface in order to determine the limit measures. We have restricted ourself to two-dimensional irrational torii, because we have no knowledge of the properties of the limit measures on other type of recurrent sets. The case of a torus is tractable because it has minimal flows [27]. Our assumptions on $b$ imply that its flow on the torus is minimal. On the other hand, no other compact surface admits minimal flows (the standard argument is coming from the Euler-Poincaré characteristics).

Also it is conceivable that limit measures on general surfaces for general flows could be concentrated on proper subsets of the surface. These subsets could be recurrent subsets of the restriction of $b$ and this in spite of the fact that the topological pressure is achieved on the surface.

### 5.2 Fundamental propositions

We have

**Proposition 6** The topological pressure of a two irrational torus satisfying assumptions IV is given by

$$TP = \int_\tau cdH - trB_s,$$

where $dH$ is the unique probability measure on the torus invariant by the flow of the restriction of $b$ to $\mathbb{T}$, $B_s$ is the stable component of the restriction of the field transverse to the torus.

**Proof.**
Let us recall the formula [31]:

\[ TP = \sup \left\{ h_\nu + \int (c + \frac{D J^s}{Dt} \bigg|_{t=0}) d\nu | \nu \text{ probability measure on } \mathbb{T} \text{ invariant by the flow} \right\} \]

where \( J^s \) is the restriction of the Jacobian to the stable manifold along \( \mathbb{T} \), \( h_\nu \) is the entropy, equal to zero [41]. In the present case, since the restriction of the flow to \( \mathbb{T} \) is ergodic, there is only one invariant probability measure \( H \) and \( dH = d\theta^1 \wedge d\theta^2 \approx \int_{\mathbb{T}} d\theta^1 \wedge d\theta^2 \).

Moreover, using the notation of paragraph 1.1, \( \frac{D J^s}{Dt} \bigg|_{t=0} = -\text{tr}B_s \).

The precise characterization of the concentration of the sequence \( v_\epsilon \) is obtained by studying the renormalized sequence \( w_\epsilon \), centered at a maximum sequence point, which concentrates on a torus. In the normal coordinate system \((x', \theta_1, \theta_2)\) along the torus, defined in section 1.1-IV, the function \( w_\epsilon \) is given by

\[ w_\epsilon(x', \theta_1, \theta_2) = \frac{v_\epsilon(\sqrt{\epsilon}x', \theta_1, \theta_2)}{\bar{v}_\epsilon}. \]

We have

**Proposition 7** If a function \( w \) is a limit of a sequence \( w_\epsilon \) as \( \epsilon \) tends to zero, then the convergence is uniform on any compact set of \( \mathbb{R}^{m-2} \times \mathbb{T}_f \), where \( \mathbb{T}_f \) is the flat two-dimensional torus.

If \( \Delta_{m-2} \) denotes the Laplacian on \( \mathbb{R}^{m-2} \), then \( w \) is a weak solution of the following equation

\[
\Delta_{m-2} w + \sum_{i,j} \Omega_{ij} x^j \frac{\partial w}{\partial x^i} + (\Omega'/\nu, \nabla w) + \left[ c(0, \theta_1, \theta_2) + \frac{\Delta g \mathcal{L}}{2} + \Psi(x') \right] w = \mu w, \quad 0 < w \leq 1,
\]

where \( \tilde{b} = (k_1, k_2) \), \( x' \) are the transverse coordinates, \( \Omega_{ij} \) is the transversal part of the field \( \Omega \) and \( \mu \) is the first eigenvalue of the operator on the left hand side.

The proof will be given later. The most important result is the following

**Proposition 8** The function \( w \) of proposition 7 is in fact regular and can be written as the product of two functions in the variables \( x'' \) and \( \theta' \) respectively

\[ w(x) = w_T(x') f_T(\theta'), \]

where \( w_T \) and \( f \) satisfy

\[
\Delta_{m-2} w_T + \sum_{i,j} \Omega_{ij} x'^j \frac{\partial w_T}{\partial x'^i} + \Psi(x') w_T = \mu_1 w_T \text{ on } \mathbb{R}^{m-2} \quad (82)
\]
\[
(\Omega^\\prime, \nabla f_T) + c(\theta')f_T = \mu_2 f_T \\
\mu_1 + \mu_2 = \mu,
\]
(83)

where
\[
\mu_1 = -\text{tr}(B_s) \\
\mu_2 = \int_T c dH.
\]
(85)

is the first eigenvalue of the operator \(\Delta_{m-2} + \sum_{i,j} \Omega_{ij}x^i \frac{\partial}{\partial x^j} + \Psi(x')\) and

Finally, using the blow-up analysis, we can characterize the limit measures along the torus by

**Proposition 9** If \(\nu\) denotes a limit measure on a torus \(\mathbb{T}\), then it is absolutely continuous with respect to the invariant measure \(d\theta_1 d\theta_2\). If \(h_T(\theta)\) is the density of \(\nu\) with respect to the probability measure invariant by the flow, we have

\[
h_T(\theta) = \frac{\gamma^2 \int_{\mathbb{R}^{n-2}} w^2_T(\tilde{x}'') d\tilde{x}''}{\sum_{T' \in S'} \gamma^2 \int_{T'} \int_{\mathbb{R}^{n-2}} w^2_T f^2_T(\theta) dS_{T'} f^2_T(\theta)},
\]

where \(w\) is the function defined by proposition 7, \(\gamma\) is the modulating coefficient and \(S'\) is the subset of tori, where the topological pressure is achieved.

**Proof.** By definition of the concentration coefficient (see definition 2), Proposition 9 is a consequence of Proposition 8, the \(L^2\) convergence of the blow up sequence \(w_\epsilon\) and Fubini’s theorem. The convergence of the sequence \(w_\epsilon\) in \(L^2\) can be proved following the steps of section 4.3.1: In a tubular neighborhood \(T(\delta)\) of the \(\mathbb{T}\), there exists constants \(C > 0\) and \(C_0 > 0\), such that

\[
w_\epsilon(x'', \theta_1, \theta_2) \leq C e^{-C_0 \frac{\text{dist}(x, \mathbb{T})^2}{\epsilon}} \text{ for } x \in T(\delta).
\]

The limit measure \(h_T(\theta)\) restricted to a torus is computed using a regular solution of a transport equation that we shall describe now.

**Lemma 15** On an irrational torus \(\mathbb{T}\) endowed with the coordinate system 1.1, consider a \(C^\infty\) function \(c\) and the field

\[
\Omega^\\prime = k_1 \frac{\partial}{\partial \theta_1} + k_2 \frac{\partial}{\partial \theta_2},
\]
(87)
where \( k_1 \) and \( k_2 \) are defined in 1.1. If the small divisor assumption (5) is satisfied, then the space of regular and bounded solutions \( f_T \) of the transport equation

\[
< \Omega^\prime, \nabla f_T > + c(\theta_1, \theta_2) f_T = \mu_2 f_T \tag{88}
\]
is of dimension one and necessarily

\[
\mu_2 = \int_T c(\theta_1, \theta_2) dH.
\]

**Proof.**
The existence of a solution of 88 is obtained by developing \( c \) in Fourier series. Writing \( w = e^h \),

\[
< \Omega^\prime, \nabla h > = \lambda - c(\theta_1, \theta_2). \tag{89}
\]

We look for \( h \) as a Fourier series

\[
\sum_{m_1, m_2} h_{m_1, m_2} e^{i(k_1 m_1 + k_2 m_2)}
\]

Because \( c \) is \( C^\infty \), it can be expanded in Fourier series

\[
c(\theta_1, \theta_2) = \sum_{m_1, m_2} c_{m_1, m_2} e^{i(k_1 m_1 + k_2 m_2)},
\]

where the coefficients \( c_{m_1, m_2} \) are rapidly decreasing. From equation (88) the coefficients are given by

\[
h_{m_1, m_2} = i \frac{c_{m_1, m_2}}{m_1 k_1 + m_2 k_2}, \text{ for } (m_1, m_2) \neq (0, 0).
\]

The compatibility condition coming from the topological pressure insures that \( \mu_2 = c_{0,0} = \int_T c(\theta_1, \theta_2) dH \), which is exactly the Fourier coefficient. It is well known that the small divisor condition (5) implies that the sequence \( h_{m_1, m_2} \) is rapidly decreasing and thus \( h \) exists, is a regular function and so is \( w = e^h \). We remark that since \( h \) is defined up to an additive constant, the space of \( C^\infty \) solutions of equation (88) is of dimension of 1.

**Remark 1.** The value of \( \mu_2 \) can also be obtained by simple considerations: the function \( f_T \) solution of equation (88) is given by the formula

\[
f_T(X(t)) = f_T(X(0)) e^{\mu_2 t - \int_0^t c(X(s)) ds}
\]
along any trajectory $X(t)$ of the restriction of $b$ to the torus. The function $f_{\mathbb{T}}$ is positive on the Torus, because the trajectory $X(t)$ is everywhere dense. Also, there exists a sequence $t_n \to +\infty$ such that $X(t_n) \to X(0)$, hence
\[
\lim_{n \to \infty} e^{\mu_2 t_n - f_{\mathbb{T}}(t_n) c(X(s))} = 1.
\]
This implies that
\[
\mu_2 = \lim_{n \to \infty} \frac{1}{t_n} \int_0^{t_n} c(X(s))ds.
\]
Using the ergodic property of the trajectories, we get
\[
\lim_{t \to \infty} \frac{1}{t} \int_0^t c(X(s))ds = \int_\mathbb{T} c dH = \mu_2.
\]

5.3 Proofs of Proposition 7 and 8

We have shown on several occasions that the sequence $v_2^2 dV_g$ concentrates on the set $Z_\Psi = \{ x \in V, |\Psi_L(x) = 0 \}$. Moreover the sequence $v_\epsilon$ converges uniformly to zero on any compact set $K \subset V - Z_\Psi$. The blow up function
\[
w_\epsilon(x'', \theta_1, \theta_2) = \frac{v_\epsilon(\sqrt{\epsilon} x'', \theta_1, \theta_2)}{\bar{v}_\epsilon},
\]
satisfies in a tubular neighborhood $T_\mathbb{T}(\delta)$ of the torus $\mathbb{T}$, the following equation
\[
\Delta_m - 2 \epsilon w_\epsilon + \frac{\Omega_1(\sqrt{\epsilon} x'')}{\sqrt{\epsilon}} w_\epsilon + (c + \frac{\Delta g L}{2} + \frac{\Psi_L(\sqrt{\epsilon} x'')}{\epsilon}) w_\epsilon + R_\epsilon(x'', \theta_1, \theta_2) = \lambda_\epsilon w_\epsilon, \tag{90}
\]
where
\[
R_\epsilon = \sqrt{\epsilon} L_1(w_\epsilon) + \epsilon L_2(w_\epsilon) + \epsilon Q(w_\epsilon),
\]
and $\Delta_m^\epsilon$ is the Laplacian in the $x''$-variables only, converging uniformly to the standard Euclidean Laplacian operator on the class of functions with compact support defined on $\mathbb{R}_m$. $L_1$ and $L_2$ are first order operators containing Christoffel symbols and $\Omega//\$ is defined in (87).

As $\epsilon$ goes to zero, every limit $w$ of $w_\epsilon$ satisfies in the weak sense the following equation (the argumentation is similar to the one provided for limit cycles)
\[
\Delta_m - 2 w + \Omega//, \nabla w > + \sum_{i,j} \Omega_{ij} x^j \frac{\partial w}{\partial x^i} + (c(0, \theta_1, \theta_2) +
\]
\[
\frac{\Delta g L}{2} |_{\mathbb{T}} + \psi_2(x'') w = \mu w \text{ on } \mathbb{R}^{n-2} \times \mathbb{T}, \tag{91}
\]
where
\[ \mu = \lim_{n \to \infty} \lambda_n. \] (92)

It is understood that this equation is defined on the universal covering \( \mathbb{R}^{n-2} \times \mathbb{R}^2 \) of the normal bundle of \( \mathbb{T} \) in \( V \). \( \psi_2 \) is polynomial of order 2. Consider
\[ \tilde{w}(x, \theta) = \frac{w(x, \theta)}{f(x, \theta)}, \]
where the function \( f(x, \theta) \) is defined in Lemma 15, then
\[ \Delta^m \tilde{w} + \sum_{i,j} \Omega_i \Omega_j \frac{\partial \tilde{w}}{\partial x_i} \left( \frac{\Delta \mathcal{L}}{2} + \psi_2(x') \right) \tilde{w} = \mu_1 \tilde{w}, \] (93)
where \( \mu_1 = -tr B_s \), this relation follows from the definition of the Topological Pressure. Along the characteristics, \( (\dot{\theta}_1 = k_1, \dot{\theta}_2 = k_2) \), we define
\[ \tilde{w}(x', t) = \tilde{w}(x'', \theta_1(t), \theta_2(t)) \] (94)
then
\[ \Delta^m \tilde{w} + \sum_{i,j} \Omega_i \Omega_j \frac{\partial \tilde{w}}{\partial x_i} \left( \frac{\Delta \mathcal{L}}{2} + \psi_2(x'') \right) \tilde{w} = \mu_1 \tilde{w}. \] (95)

Because \( \tilde{w} \) is a solution of a parabolic equation, by the regularity theorems (see the case of limit cycles), we conclude that the solution is regular. Moreover on each compact set of \( \mathbb{R}^{n-2} \times \mathbb{R}^2 \), \( w \) is the uniform limit of a sequence \( w_n \), where \( \epsilon_n \) goes to 0. To derive an explicit expression for the function \( \tilde{w} \) defined by equation (95), we use Kolmogorov’s representation formula. Define
\[ \tilde{z}(x', t) = \tilde{w}(x', t) \exp \left( \frac{1}{2} < Ax', x' >_{\mathbb{R}^{m-2}} \right) \]
then
\[ \tilde{z}(x', t) = \frac{e^{-tr B_s}}{(4\pi)^{\frac{m-1}{2}} \sqrt{\det Q_t}} \int_{\mathbb{R}^{m-1}} \tilde{w}(y, \theta_1, \theta_2) e^{-\eta(x,y,t)} dy. \] (96)
This notation have been introduced in lemma 7. \( \tilde{w} \) is regular and bounded, thus we conclude as in lemma 7 that \( \tilde{z} \) is well defined and belongs to Tychonoff’s class. Because \( \tilde{z} \) is not a periodic function of the variable \( t \), we cannot use lemma 10 to conclude. We have to modify slightly the proof of lemma 7 to obtain an explicit expression of the function \( \tilde{z} \). Instead of periodicity, we use the density of the flow of \( \Omega_\parallel \) on the torus: We denote by \( \tau(t, \theta_1, \theta_2) \) the flow of \( \Omega_\parallel \). For any point \( (\theta_1, \theta_2) \), there exists a sequence \( t_n \) converging to infinity such that \( \tau(t_n, \theta_1, \theta_2) \to (\theta_1, \theta_2) \). For \( n \) sufficiently large,
\[ \int_{\mathbb{R}^{m-2}} \tilde{w}((\eta' + R_{t_n}^{-1} P_{t_n} x'), \theta_1, \theta_2) e^{-\frac{1}{2} ||R_{t_n} \eta'||_{\mathbb{R}^{m-2}}^2 - \frac{1}{2} ||R_{t_n} \eta'||_{\mathbb{R}^{m-1}}^2 d\eta'} \leq \frac{1}{n}. \]
Here, we have used relation (33). Using the continuity of $\tilde{w}$ and letting $n$ go to infinity, we get

$$\tilde{z}(x', \theta_1, \theta_2) = \exp\left[ -\frac{1}{4} \left( \int_0^{+\infty} e^{tB_s} e^{tB_s^*} dt \right)^{-1} x_s, x'_s \right] \tilde{\chi}(\theta_1, \theta_2),$$

where

$$\tilde{\chi}(\theta_1, \theta_2) = \int_{\mathbb{R}^{m-2}} \tilde{w}(\eta', \theta_1, \theta_2) e^{-\frac{1}{2} ||R_s, \eta'_s||^2_{\mathbb{R}^{m-2}} - \frac{1}{2} ||R_u, \eta'_u||^2_{\mathbb{R}^{m-2}}} d\eta'.$$

This identity proves that $\tilde{w}$ is the product of two functions: an exponential function depending only on the variable $x_s$ and a function $\tilde{\chi}$ of the variable $(\theta_1, \theta_2)$.

Finally, setting $M_s = \int_{\mathbb{R}^{m-2}} e^{tB_s} e^{tB_s^*} dt$, (97)

$$w(x', \theta_1, \theta_2) = \tilde{\chi}(\theta_1, \theta_2) f_T(\theta_1, \theta_2) \exp\left[ -\left( \frac{1}{2} < Ax', x'_s >_{\mathbb{R}^{m-2}} - \frac{1}{4} M_s^{-1} x_s, x'_s >_{\mathbb{R}^{m-2}} \right) \right].$$

Because $w$ satisfies equation (82) and $f_T$ equation 83, $\tilde{\chi}$ satisfies the equation

$$< \Omega^{\perp}, \nabla \tilde{\chi} >= 0.$$

Since $\Omega^{\perp}$ is ergodic, we conclude that $\tilde{\chi}$ is a constant. □

**Proof of proposition 9**

The proof uses the explicit expression of the function $w$ given by expression (98). Indeed, following the steps of section 4.4, it can be proved that there exists a sequence $w_{\epsilon_n}$ which converges to $w$ in $L_2(\mathbb{R}^{m-2} \times T)$. Using propositions 7 and 8 we have

$$1 = \int_{V_n} v_{\epsilon_n}^2 = \sum_{T} \hat{v}_{\epsilon_n}^2 \epsilon_n^{(m-2)/2} \int_{B_T(\delta/\epsilon_n)} w_{\epsilon_n}^2(\sqrt{\epsilon_n} x) dV_{g_{\epsilon_n}} + \hat{v}_{\epsilon_n}^2 \epsilon_n^{(m-1)/2} \sum_{T} \int_{T_T(\delta/\epsilon_n)} w_{\epsilon_n}^2(\sqrt{\epsilon_n} x', \theta) dV_{g_{\epsilon_n}} + \hat{v}_{\epsilon_n}^2 \epsilon_n^{(m-2)/2} \sum_{T} \int_{T_T(\delta/\epsilon_n)} w_{\epsilon_n}^2(\sqrt{\epsilon_n} x', \theta_1, \theta_2) dV_{g_{\epsilon_n}}.$$

Now suppose that a torus is charged,

$$\lim_{\epsilon_n \to 0} \hat{v}_{\epsilon_n}^2 \epsilon_n^{(m-2)/2} = C > 0$$
and in that case,
\[ \lim_{\epsilon_n \to 0} \bar{v}_{\epsilon_n}^{m-1}/2 = \lim_{\epsilon_n \to 0} \bar{v}_{\epsilon_n}^{m}/2 = 0, \]
thus no cycles or points can contribute to the limit measure. Using the exponential decay of the sequence \( w_{\epsilon_n} \) and the strong convergence in \( L_2(\mathbb{R}^{n-2} \times T) \) to the function \( w \), we obtain the following expression for the constant \( C \):
\[
1 = C \sum_{T} \int_{T} \int_{\mathbb{R}^{n-2}} \gamma_T^2 w_T(x')^2 f_T(\theta_1, \theta_2) dxd\theta_1 d\theta_2,
\]
where the sum is extended to the charged torii, where the topological pressure is attained. \( \gamma_T \) is a modulating coefficient. When no torii are charged, the limit measures have been studied in the section 4 on limit cycles. 

6 Conjectures and open questions

In this section, we offer some conjectures and state some open problems.

An interesting extension of the present work would be to study the blow up function on a component of the recurrent set that is a manifold \( M_k \) of dimension \( k \) (\( k \geq 2 \)) normally hyperbolic. Probably to obtain substantial results, one has to make more assumptions on the hyperbolic structure and the restriction of the field to the manifold, such as assuming that the restriction is ergodic (with respect to some invariant measure on the manifold).

For example, it could happen that with some unspecified assumptions on the field and on \( M_k \), we could have a variable-separation phenomenon for the limit of the blown up sequence as in lemma 15 for the case of a torus and in proposition 3 for a cycle.

In a different direction, if the restriction of the field to the manifold \( M_k \) has a nonzero entropy, it would be interesting to study the limit of the blown up function, as we did here in lemma 15 when the entropy is zero. In particular, one can expect a new characterization of the entropy \( \mu \) as follow:
\[
\mu = \lim_{T \to \infty} \frac{1}{T} \ln w_{M_k}(X(T)),
\]
where \( X(T) \) is the flow of the field restricted to \( M_k \), \( w_{M_k} \) is a nonzero solution of equation
\[
< \Omega^//, \nabla w_{M_k} > + cw_{M_k} = \mu_2 w_{M_k},
\]
\( \Omega^// \) is the restriction of the field to \( M_k \) and
\[
\mu_2 = \mu + \int c dH.
\]
\( H \) is the unique invariant and ergodic measure with respect to \( \Omega^// \).
6.1 Anisotropic concentration when $\Psi$ vanishes to order 4 and more.

In the present work, we have assumed that the special Lyapunov function $\Psi_L$ vanishes at order 2 on the recurrent set of the field. However, when $\Psi_L$ vanishes to a higher order (four or more), the analysis used here to study the concentration process of the eigenfunction sequence does not apply anymore: when $\Psi_L$ vanishes at order 4, the limit measures are not necessarily concentrated on the components of the recurrent set on which the topological pressure is attained. In fact, the sequence $w_\epsilon$ may not converge in $L^2$ because in equation (98), the function $\psi_2$ is identically zero. To analyze furthermore this case, we consider a small ball centered at a saddle point $S$ where the dimension of the stable and unstable spaces satisfies $m_s + m_u = m$. Consider a canonical cartesian structure in the neighborhood of a saddle point, defined by the stable and unstable manifolds [37]. In that cartesian structure $(x_s$ (stable), $x_u$ (unstable)), define the following cones: for small $\delta > 0$,

$$C_s(\delta) = \{x = (x_u, x_s) \text{ such that } |x_u| \leq |x_s| \leq \delta\}$$

and

$$C_u(\delta) = \{x = (x_u, x_s) \text{ such that } |x_s| \leq |x_u| \leq \delta\}.$$ 

We have the following: for small $\delta > 0$,

$$\lim_{\epsilon \to 0} \frac{1}{\epsilon^{m/2} v_\epsilon^2} \int_{C_s(\delta)} v_\epsilon^2 \, dx = \lim_{\epsilon \to 0} \frac{1}{\epsilon^{m/2} v_\epsilon^2} \int_{B_u(0,\sqrt{\epsilon})} \int_{B_u(0,||x_s||)} w_s^2(x_s) \, dx^u \, dx^s < \infty,$$ 

where the exterior integral converges because $\int_{B_u(0,||x_s||)} \, dx^u$ is bounded by a polynomial in $||x_s||$ and $w_s^2$ decays exponentially.

Now,

$$\frac{1}{\epsilon^{m/2} v_\epsilon^2} \int_{C_u} v_\epsilon^2 = \int_{B_u(0,\sqrt{\epsilon})} \int_{B_s(0,||x_u||)} w_s^2(x_s) \, dx^u \, dx^s \geq C \epsilon^{-n_u/2} \delta^{n_u},$$

where $C > 0$ is a constant. Thus the ratio in equation 102 converges to zero.
6.2 Final remarks

6.2.1 Concentration near polycycle

When the recurrent set of the field contains polycycles, (that is a close curve which is the union of critical points and separatrices joining these points). In that case, we expect that the concentration will not occur uniformly on the polycycle, but rather at the critical points. It is unclear what is the effect on the concentration phenomena of the discontinuity of the tangent vectors at the critical points.

6.2.2 The case of Hamiltonian systems

We have developed here a method for hyperbolic field, but for Hamiltonian systems, the present method cannot be applied and a new approach has to be introduced. In particular it is not clear what is the support of the limit measures.

6.2.3 Study the spectrum

The characterization of the set of possible limit measures of eigenfunctions associated to other eigenvalues remains an open problem. Such a study is feasible because for a Morse-Smale drift, the blow up analysis near the points and cycles leads to an eigenfunction problem of the Ornstein-Ulhenbeck operator and the solution can be expressed explicitly by Hermite functions.

6.2.4 Asymptotic computation

The existence of a asymptotic expansion,

$$\lambda_\epsilon \approx \text{topological pressure} + I_1 \epsilon^{1/2} + I_2 \epsilon + ...$$

of the first eigenvalue $\lambda_\epsilon$ as a function of $\epsilon$ is an open problem and so is the determination of the coefficients $I_k$.

6.3 Appendix

1) The notations are those of the section “Proofs of the theorems” except we denote the function $w$ by $w_0$ so as to avoid confusions. First let us show that $w_0$ is $C^\infty$. Recall that $w_0$ belongs to $L^2_{loc}$ and that it is a weak solution of equation

$$- \sum_{i=1}^{m-1} \frac{\partial^2 w}{(\partial x_i)^2} + \sum_{i,j=1}^{m-1} \Omega^i_j x^j \frac{\partial w}{\partial x_i} + \frac{\partial w}{\partial \theta} + (c + \frac{\Delta_2 \mathcal{L}}{2})(0, \theta) + \psi_2(x')w = \lambda w$$

The theorems 13.4.1, page 191, vol.II and 4.4.1, page 110, vol I of ([26]) show that any weak solution of such a parabolic operator is $C^\infty$. 

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2) Recall the operator:

\[
L_\varepsilon = L_{2\varepsilon} + L_{1\varepsilon} + L_{0\varepsilon}
\]

where:

\[
L_{2\varepsilon} = -\sum_{i,j=1}^{m-1} g^{ij}_\varepsilon \frac{\partial^2}{\partial x^i \partial x^j} - 2 \sqrt{\varepsilon} \sum_{i=1}^{m-1} g^{im}_\varepsilon \frac{\partial^2}{\partial x^i \partial \theta} - \varepsilon g^{mm}_\varepsilon \frac{\partial^2}{\partial \theta^2}
\]

\[
L_{1\varepsilon} = \sum_{k=1}^{m-1} \left( \frac{\Omega^k}{\sqrt{\varepsilon}} - \sqrt{\varepsilon} \sum_{i,j=1}^{m} g^{ij}_\varepsilon \Gamma^k_{\varepsilon ij} \right) \frac{\partial}{\partial x^k} + \left( \Omega^m - \varepsilon \sum_{i,j=1}^{m} g^{ij}_\varepsilon \Gamma^m_{\varepsilon ij} \right) \frac{\partial}{\partial \theta}
\]

\[
L_{0\varepsilon} = c_\varepsilon + \frac{(\Delta_\varepsilon L)_\varepsilon}{2} + \frac{\Psi_\varepsilon}{\varepsilon}
\]

and 
\[
g^{ij}_\varepsilon = g^{ij}(x'\sqrt{\varepsilon}, \theta), 1 \leq i, j \leq m, \quad \Gamma^k_{\varepsilon ij} = \Gamma^k_{ij}(x'\sqrt{\varepsilon}, \theta), \quad (\Delta_\varepsilon L)_\varepsilon = \Delta_\varepsilon L(x'\sqrt{\varepsilon}, \theta), \quad c_\varepsilon = c(x'\sqrt{\varepsilon}, \theta), \quad \Psi_\varepsilon = \Psi_L(x'\sqrt{\varepsilon}, \theta).
\]

Using Schweins\' formulas we can write:

\[
L_\varepsilon = L'_{2\varepsilon} + L'_{1\varepsilon} + L_{0\varepsilon}
\]

\[
L'_{2\varepsilon} = \sum_{k=1}^{m} X^*_k X_k
\]

\[
L'_{1\varepsilon} = \sum_{k=1}^{m-1} \left( \frac{\Omega^k}{\sqrt{\varepsilon}} - \sqrt{\varepsilon} \sum_{i,j=1}^{m} g^{ij}_\varepsilon \Gamma^j_{\varepsilon ji} \right) \frac{\partial}{\partial x^k} + \left( \Omega^m + \varepsilon \sum_{i,j=1}^{m} g^{ij}_\varepsilon \Gamma^j_{\varepsilon ji} \right) \frac{\partial}{\partial \theta}
\]

where:

\[
X_k = \sum_{i=1}^{k} a^i_k \frac{\partial}{\partial x^i}
\]

for \(k = 1, \ldots, m-1\),

\[
X_m = \sum_{i=1}^{m-1} a^i_k \frac{\partial}{\partial x^i} + \sqrt{\varepsilon} a^m_k \frac{\partial}{\partial \theta}.
\]
$X^*_k$ is the formal adjoint of $X_k$

$$X^*_k = -\sum_{i=1}^{k} \frac{\partial}{\partial x^i} a^i_k,$$

for $k=1,\ldots,m-1$, and

$$X^*_m = -\sum_{i=1}^{m-1} \frac{\partial}{\partial x^i} a^i_k - \sqrt{\varepsilon} \frac{\partial}{\partial \theta} a^m_k,$$

where $1 \leq i \leq j \leq m$:

$$a^i_j = \frac{G(ij)}{\sqrt{G(jj)G(j+1j+1)}},$$

and

$$G(jj) = \text{det}\{g^{\alpha\beta}; j \leq \alpha, \beta \leq m\}$$

$$G(ij) = \text{det}\{g^{\alpha\beta}; i \leq \alpha \leq m, j \leq \beta \leq n\}.$$

$$g^{\alpha\beta}(x', \theta) = g^{\alpha\beta}(\sqrt{\varepsilon}x', \theta)$$

When $\varepsilon \to 0+$, $L_\varepsilon \to L_0 = L'_{20} + L'_{10} + L_{00}$ where:

$$L'_{20} = -\sum_{i=1}^{m-1} \frac{\partial^2}{(\partial x^i)^2}$$

$$L'_{10} = \sum_{i,j=1}^{m-1} \Omega^i_j x^j \frac{\partial}{\partial x^i} + \frac{\partial}{\partial \theta}$$

Recall that by our choice of coordinates along a cycle $\Omega^m(0, \theta) = 1$. Also $X_k \to \frac{\partial}{\partial x^k}$ as $\varepsilon \to 0$, if $1 \leq k \leq m-1$, and $X_m \to 0$.

We know that $w$ is a weak solution of the equation:

$$L_0 w = \lambda w \quad (103)$$

where $\lambda = \lim_{\varepsilon \to 0} \lambda_\varepsilon$. Hence it follows from Theorem 22.1, page 353, in volume 274 of ([26]) that $w$ is a $C^\infty$ solution of equation (103).

On the other hand for any $\varepsilon > 0$, $w_\varepsilon$ satisfies the relation

$$L_\varepsilon w_\varepsilon = \lambda_\varepsilon w_\varepsilon \text{ on } \mathbb{R} \times B_0^{m-1}(\delta/\sqrt{\varepsilon})$$
Let $\delta_\varepsilon = w_\varepsilon - w_0$. Then $\delta_\varepsilon$ satisfies the following equation on $\mathbb{R} \times B_{0}^{m-1}(\delta/\sqrt{\varepsilon})$

$$L_\varepsilon \delta_\varepsilon = \lambda_\varepsilon \delta_\varepsilon + (L_0 - L_\varepsilon) w_0 + (\lambda_\varepsilon - \lambda_0) w_0$$

(104)

It is easy to see that for any compact $K \subset \mathbb{R}^{m-1}$, there is an $\varepsilon_K > 0$ such that $K \subset B_{0}^{m-1}(\delta/\sqrt{\varepsilon})$ if $0 < \varepsilon \leq \varepsilon_K$. Moreover there exists a $C^\infty$ differential operator $P_\varepsilon$ defined on $\mathbb{R} \times B_{0}^{m-1}(\delta/\sqrt{\varepsilon})$ such that

$$L_\varepsilon - L_0 = \sqrt{\varepsilon}P_\varepsilon$$

Taking a compact set $K_1$, two functions $\phi_1, \phi_2$ in $C^\infty(\mathbb{R}^{m-1}; [0, 1])$ such that $\phi_1 = 1$ on $K_1$ and that $\phi_2 = 1$ on the support of $\phi_1$, for any $s > 0$, there are constants $\sigma = \sigma(K_2) > 0$, $\tau < 0$, $C_s = C(s, \phi_1, \phi_2)$ such that for all $\varepsilon$, $0 \leq \varepsilon \leq \varepsilon_K$, where $K_2$ is the support of $\phi_2$,

$$||\phi_1 \delta_\varepsilon||_{s+\sigma} \leq C_s (\varepsilon ||\phi_2 w_0||_{s+2} + ||\lambda_\varepsilon - \lambda_0|| ||\phi_2 w_0||_{s} + ||\phi_2 \delta_\varepsilon||_{1})$$

(105)

This follows from Lemmas 22.2.4, page 356 and 22.2.5, page 357 in vol. III of ([26]). Because in our case the coefficients of the operator $L_\varepsilon$ and all their derivatives are continuous functions of the variables $(x', \theta)$ and also $\varepsilon \in [0, 1]$ it is easy to check that all the constant appearing in the proofs of section 22.2 in vol. III of ([26]) can be taken independent of $\varepsilon$. $\phi_2 w_0$ is the strong limit of the sequence $\{\phi_2 w_{\varepsilon_n} | n \in \mathbb{N}\}$ in any space $H_\rho$ with $\rho < 0$.

The equation (105) implies that the sequence $\{\delta_\varepsilon | n \in \mathbb{N}\}$ and all its derived sequences tend to 0 uniformly.

### 6.4 Appendix II: estimation of $E_x\{\chi_3(X_\varepsilon)e^{-\int_0^t \frac{\Psi_\varepsilon(X_3(s))ds}{\varepsilon}}\}$

$\chi_3 = \chi_{E_3}$, $E_3 = \{\gamma|d_\infty(\gamma, \gamma_x) \leq \eta\}$. Evaluate: $E_x[\chi_3(X_\varepsilon(t)) \exp\left(-\int_0^t \frac{\Psi_\varepsilon(X_3(s))ds}{\varepsilon}\right)]$

$$dX_\varepsilon(t) = -\Omega_\varepsilon(X_\varepsilon(t))dt + \sqrt{2\varepsilon}\sigma(X_\varepsilon(t))dw(t),$$

(106)

where

$$\Omega_\varepsilon(x) = \Omega(x) + \varepsilon \hat{\Omega}(x), \quad \text{where}$$

$$\hat{\Omega}^k = \sum_{ij=1}^m g^{ij} \Gamma_{ij}^k, 1 \leq k \leq m.$$  

(107)

Define

$$Y_\varepsilon = X_\varepsilon - \gamma_x,$$

(109)

d$$Y_\varepsilon(t) = -(\gamma'_x + \Omega_\varepsilon(Y_\varepsilon(t) + \gamma_x(t))) dt + \sqrt{2\varepsilon}\sigma(Y_\varepsilon(t) + \gamma_x(t))dw(t)$$

(110)

Define

$$dZ_\varepsilon(t) = (\Omega(\gamma_x(t)) - \Omega_\varepsilon(Z_\varepsilon(t) + \gamma_x(t))) dt + \sqrt{2\varepsilon}\sigma(Z_\varepsilon(t) + \gamma_x(t))dw(t)$$

(111)
Applying Girsanov's formula between $Y_\varepsilon$ and $Z_\varepsilon$, we get

$$
\frac{dP_{Y_\varepsilon}}{dP_{Z_\varepsilon}} = \exp\left\{-\int_0^t \frac{1}{2} \left( \gamma'_x(s) + \Omega_x(\gamma_x(s)) \right) ds - \Omega_x(Z_\varepsilon(s)) - \gamma_x(s) + \frac{1}{2} \int_0^t \left| \gamma'_x(s) + \Omega_x(\gamma_x(s)) \right|^2_{g_x(s) + \gamma_x(s)} \right\}
$$

$$
E_x[\chi_3(X_\varepsilon(t))] \exp\left(-\int_0^t \frac{\Psi_x(X_\varepsilon(s))}{\varepsilon} ds\right) = E_{X_\varepsilon}^P[\chi_3(Z_\varepsilon(t) + \gamma_x(t)) \exp(-I(t))]
$$

$$
I(t) = \frac{1}{\varepsilon} \int_0^t \Psi_x(X_\varepsilon(s) + \gamma_x(s)) + \frac{1}{2} \int_0^t \left| \gamma'_x(s) + \Omega_x(\gamma_x(s)) \right|^2_{g_x(s) + \gamma_x(s)} ds
$$

$$
Z_\varepsilon = \sqrt{\varepsilon}z_1 + \varepsilon z_2 + \varepsilon^{\frac{3}{2}}R_{3,\varepsilon} = \sqrt{\varepsilon}z_1 + \varepsilon R_{2,\varepsilon} = \sqrt{\varepsilon}R_{1,\varepsilon}
$$

$$
dz_1 = -\frac{\partial \Omega}{\partial x}(\gamma_x)z_1 dt + \sqrt{2}\sigma(\gamma_x) dw, \tag{112}
$$

$$
dz_2 = \left[ -\frac{\partial \Omega}{\partial x}(\gamma_x)z_2 + \frac{1}{2} \frac{\partial^2 \Omega}{(\partial x)^2}(\gamma_x) [z_1 \otimes R_{2,\varepsilon} + \sqrt{\varepsilon}R_{2,\varepsilon} \otimes R_{2,\varepsilon}] ight] dt + \sqrt{2} \frac{\partial \sigma}{\partial x}(\gamma_x) [z_1] dw \tag{113}
$$

where $[z_1, z_1]$ mean the tensorial product. We define the following notation: if $A$ and $B$ denote two n-dimensional vectors, $A \otimes B$ is a matrix of coordinates:

$$(A \otimes B)_{kq} = \frac{1}{2} (A_k B_q + A_q B_k)$$

$$
dR_{3,\varepsilon} = \left\{ -\frac{\partial \Omega}{\partial x}(\gamma_x) R_{3,\varepsilon} - \frac{\partial^2 \Omega}{(\partial x)^2}(\gamma_x) [z_1 \otimes R_{2,\varepsilon} + \sqrt{\varepsilon}R_{2,\varepsilon} \otimes R_{2,\varepsilon}] ight\} dt
$$

$$
- \int_0^1 \frac{1}{2} \frac{\partial^3 \Omega}{(\partial x)^3}(\gamma_x + \theta Z_\varepsilon) [R_{1,\varepsilon} \otimes R_{1,\varepsilon} \otimes R_{1,\varepsilon}] d\theta + \int_0^1 \frac{\partial \hat{\Omega}}{\partial x}(\gamma_x + \theta Z_\varepsilon) [R_{1,\varepsilon}] d\theta dt
$$

$$
+ \sqrt{2} \frac{\partial \sigma}{\partial x}(\gamma_x) [R_{2,\varepsilon}] dw + \int_0^1 (1 - \theta) \frac{\partial^2 \sigma}{(\partial x)^2}(\gamma_x + \theta Z_\varepsilon) [R_{1,\varepsilon} \otimes R_{1,\varepsilon}] d\theta dw
$$

$$
dR_{2,\varepsilon} = \left\{ -\frac{\partial \Omega}{\partial x}(\gamma_x) R_{2,\varepsilon} - \frac{1}{2} \frac{\partial^2 \Omega}{(\partial x)^2}(\gamma_x) [R_{1,\varepsilon} \otimes R_{1,\varepsilon}] ight\} dt
$$

$$
\sqrt{\varepsilon} \int_0^1 \frac{1}{2} \frac{\partial^3 \Omega}{(\partial x)^3}(\gamma_x + \theta Z_\varepsilon) [R_{1,\varepsilon} \otimes R_{1,\varepsilon} \otimes R_{1,\varepsilon}] d\theta + \int_0^1 \frac{\partial \hat{\Omega}}{\partial x}(\gamma_x + \theta Z_\varepsilon) [R_{1,\varepsilon}] d\theta dt
$$

$$
+ \sqrt{2} \frac{\partial \sigma}{\partial x}(\gamma_x) [R_{2,\varepsilon}] dw + \int_0^1 (1 - \theta) \frac{\partial^2 \sigma}{(\partial x)^2}(\gamma_x + \theta Z_\varepsilon) [R_{1,\varepsilon} \otimes R_{1,\varepsilon}] d\theta dw
$$

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Let us decompose

\[ I(t) = \frac{1}{\varepsilon} I_0(t) + I_1(t) + I_2(t) + I_3(t), \]

where

\[ I_0(t) = \int_0^t \left[ [\Psi_L(\gamma_\varepsilon(s))] + \frac{1}{2} \|\gamma_\varepsilon'(s) + \Omega(\gamma_x(s))\|_{\gamma(\gamma_x(s))}^2 \right] ds, \]

\[ I_1(t) = \frac{1}{\varepsilon} \int_0^t \left[ \sum_{k=1}^m \frac{\partial \Psi_L(\gamma_x)}{\partial x^k} + \frac{1}{2} \sum_{ij=1}^m \frac{\partial g_{ij}(\gamma_x)}{\partial x^k} (\gamma_x' i + \Omega^i(\gamma_x))(\gamma_x' j + \Omega^j(\gamma_x)) \right] z_1^k ds + \]

\[ \frac{\sqrt{2}}{\varepsilon} \int_0^t <\gamma_x' + \Omega_x(\gamma_x), \sigma(\gamma_x)dw>_g(\gamma_x), \]

\[ I_2(t) = \int_0^t \sum_{i,j=1}^m \frac{\partial \Psi_L(\gamma_x)}{\partial x^i} \zeta_2^i + \frac{1}{2} \sum_{i,j=1}^m \frac{\partial g_{ij}(\gamma_x)}{\partial x^i} (\gamma_x' i + \Omega^i(\gamma_x))(\gamma_x' j + \Omega^j(\gamma_x)) z_1^j \]

\[ \frac{1}{4} \sum_{ijkl=1}^m \frac{\partial^2 g_{ij}(\gamma_x)}{\partial x^k \partial x^l} (\gamma_x' i + \Omega^i(\gamma_x))(\gamma_x' j + \Omega^j(\gamma_x)) z_1^k z_1^l + \]

\[ \frac{\sqrt{2}}{\varepsilon} \int_0^t \sum_{ijkl=1}^m \left[ \frac{\partial g_{ij}(\gamma_x)}{\partial x^i} (\gamma_x' i + \Omega^i(\gamma_x)) \sigma^{ij}(\gamma_x) + g_{ij}(\gamma_x)(\gamma_x' i + \Omega^i(\gamma_x)) \frac{\partial \sigma^{ij}(\gamma_x)}{\partial x^k} \right] z_1^k dw \]

\[ I_3(t) = I_{31}(t) + I_{32}(t) + I_{33}(t) + I_{34}(t) \]

\[ I_{31}(t) = \sqrt{\varepsilon} \int_0^t \sum_{k=1}^m \left( \frac{\partial \Psi_L(\gamma_x)}{\partial x^k} + \frac{1}{2} \sum_{ij=1}^m \frac{\partial g_{ij}(\gamma_x)}{\partial x^k} (\gamma_x' i + \Omega^i(\gamma_x))(\gamma_x' j + \Omega^j(\gamma_x)) \right) R_{3,k}^k ds \]

\[ I_{32}(t) = \sqrt{\varepsilon} \int_0^t \frac{1}{2} \sum_{kl=1}^m \left( \frac{\partial^2 \Psi_L(\gamma_x)}{\partial x^k \partial x^l} + \frac{1}{2} \sum_{ij=1}^m \frac{\partial^2 g_{ij}(\gamma_x)}{\partial x^k \partial x^l} (\gamma_x' i + \Omega^i(\gamma_x))(\gamma_x' j + \Omega^j(\gamma_x)) \right) \times \]

\[ (z_1^k R_{2,k}^k + z_1^l R_{2,l}^l + \sqrt{\varepsilon} R_{2,k}^k R_{2,l}^l) ds \]

\[ I_{33}(t) = \sqrt{\varepsilon} \int_0^t \int_0^1 (1 - \theta)^2 \sum_{kl,n=1}^m \left( \frac{\partial^3 \Psi_L(\gamma_x + \theta Z_\varepsilon)}{\partial x^k \partial x^l \partial x^n} + \right) \]

\[ \frac{1}{2} \sum_{ijkl,n=1}^m \frac{\partial^3 g_{ij}(\gamma_x + \theta Z_\varepsilon)}{\partial x^k \partial x^l \partial x^n} (\gamma_x' i + \Omega^i(\gamma_x))(\gamma_x' j + \Omega^j(\gamma_x)) \times \]

\[ R_{1,k}^k R_{1,l}^l R_{1,n}^n d\theta ds \]
\[
I_{34}(t) = \sqrt{\frac{\varepsilon}{2}} \int_0^1 \int_0^t (1 - \theta) \sum_{ijkl,n=1}^m \left[ \frac{\partial^2 g_{ij}(\gamma_x + \theta Z_\varepsilon)}{\partial x^k \partial x^l} (\gamma_x^i + \Omega^i(\gamma_x)) \sigma^{jn}(\gamma_x + \theta Z_\varepsilon) + \
2 \frac{\partial g_{ij}(\gamma_x + \theta Z_\varepsilon)}{\partial x^k} (\gamma_x^i + \Omega^i(\gamma_x)) \frac{\partial \sigma^{jn}(\gamma_x + \theta Z_\varepsilon)}{\partial x^l} + \
g_{ij}(\gamma_x + \theta Z_\varepsilon)(\gamma_x^i + \Omega^i(\gamma_x)) \frac{\partial^2 \sigma^{jn}(\gamma_x + \theta Z_\varepsilon)}{\partial x^k \partial x^l} \right] R_1^k R_1^l d\omega_n d\theta
\]

Estimate of \(I_1(t)\). The Euler-Lagrange equations of the minimization problem imply that if we set, for \(1 \leq k \leq m\)

\[
p_k = \sum_{i=1}^m g_{ik}(\gamma_x)(\gamma_x^i + \Omega^i(\gamma_x))
\]

we have using equations 42, 43 of the Hamiltonian system,

\[
\frac{dp_k}{ds} = \frac{\partial \Psi}{\partial x^k} - \frac{1}{2} \sum_{ij=1}^m \frac{\partial g_{ij}(\gamma_x)}{\partial x^k} p_i p_j + \sum_{j=1}^m p_j \frac{\partial \Omega^j(\gamma_x)}{\partial x^k}
\]

Hence:

\[
\frac{dp_k}{ds} = \frac{\partial \Psi}{\partial x^k} + \frac{1}{2} \sum_{ij=1}^m \frac{\partial g_{ij}(\gamma_x)}{\partial x^k} (\gamma_x^i + \Omega^i(\gamma_x))(\gamma_x^j + \Omega^j(\gamma_x)) + \sum_{j=1}^m p_j \frac{\partial \Omega^j(\gamma_x)}{\partial x^k}
\]

This implies:

\[
I_1(t) = \frac{1}{\sqrt{\varepsilon}} \int_0^t \left[ \sum_{k=1}^m \left( \frac{dp_k}{ds} - \sum_{j=1}^m p_j \frac{\partial \Omega^j(\gamma_x)}{\partial x^k} \right) z_k^k ds + \sqrt{2} < \gamma_x' + \Omega_{\varepsilon}(\gamma_x), \sigma(\gamma_x)> dw > g(\gamma_x) \right]
\]

Using the stochastic equation for \(z_1\):

\[
I_1(t) = \frac{1}{\sqrt{\varepsilon}} \int_0^t \left[ \sum_{k=1}^m \left( \frac{dp_k}{ds} - \sum_{j=1}^m p_j \frac{\partial \Omega^j(\gamma_x)}{\partial x^k} \right) z_k^k ds + \sum_{j=1}^m p_k \left( dz_1^k + \sum_{j=1}^m \frac{\partial \Omega^j(\gamma_x)}{\partial x^j} z_1^k ds \right) \right]
\]

\[
I_1(t) = \frac{1}{\sqrt{\varepsilon}} \int_0^t \left[ \sum_{k=1}^m \frac{dp_k}{ds} z_1^k ds + \sum_{j=1}^m p_k dz_1^k \right] = \sum_{j=1}^m p_k z_1^k \bigg|_0^t
\]

But, \(z_1^k(0) = 0, 1 \leq k \leq m\) and according to the boundary conditions for the optimization problem \(p_k(t) = 0, 1 \leq k \leq m\), thus

\[
I_1(t) = 0.
\]
Estimate of $I_2(t)$. Proceeding as above:

$$
\frac{1}{\sqrt{\varepsilon}} \int_0^t \left[ \sum_{k=1}^m \frac{\partial \Psi_L(\gamma_x)}{\partial x^k} \zeta^k + \frac{1}{2} \sum_{ij=1}^m \frac{\partial g_{ij}(\gamma_x)}{\partial x^k} (\gamma_x^i + \Omega^i(\gamma_x))(\gamma_x^j + \Omega^j(\gamma_x)) \right] ds 
$$

$$
= \frac{1}{\sqrt{\varepsilon}} \int_0^t \sum_{k=1}^m \left( \int_{\Omega}^t p_k \frac{\partial \Omega^k(\gamma_x)}{\partial x^k} \right) \zeta^k dt 
$$

and using the stochastic equation for $z_2$ :

$$
\sqrt{2} \int_0^t \sum_{ijkl=1}^m g_{ij}(\gamma_x)(\gamma_x^i + \Omega^i(\gamma_x)) \frac{\partial \sigma^j(\gamma_x)}{\partial x^k} \zeta^k dw_t = 
$$

$$
\int_0^t \sum_{k=1}^m p_k \left( dz^k + \left[ \sum_{j=1}^m \frac{\partial \Omega^k(\gamma_x)}{\partial x^j} \zeta^j + \frac{1}{2} \sum_{j,l=1}^m \frac{\partial^2 \Omega^k(\gamma_x)}{\partial x^j \partial x^l} \zeta^j \zeta^l + \sum_{k=1}^m p_k \hat{\Omega}^k(\gamma_x) \right] ds 
$$

Then inserting these expressions in $I_2(t)$ :

$$
I_2(t) = \int_0^t \left( \frac{1}{2} \sum_{k=1}^m \frac{\partial^2 \Psi_L(\gamma_x)}{\partial x^k \partial x^l} \zeta^k \zeta^l + \frac{1}{2} \sum_{ij=1}^m \frac{\partial^2 g_{ij}(\gamma_x)}{\partial x^k \partial x^l} (\gamma_x^i + \Omega^i(\gamma_x))(\gamma_x^j + \Omega^j(\gamma_x)) + \sum_{j=1}^m p_j \frac{\partial^2 \Omega^j(\gamma_x)}{\partial x^k \partial x^l} \right) ds + 
$$

$$
\frac{1}{2} \sum_{ij=1}^m g_{ij}(\gamma_x)(\zeta^i + \sum_{l=1}^m \frac{\partial \Omega^i(\gamma_x)}{\partial x^l} \zeta^l + \sum_{k=1}^m \frac{\partial \Omega^i(\gamma_x)}{\partial x^k} \zeta^k) ds + 
$$

$$
+ \int_0^t \left[ \sum_{ij=1}^m \frac{\partial g_{ij}(\gamma_x)}{\partial x^k} (\gamma_x^i + \Omega^i(\gamma_x)) \left( d\zeta^j + \sum_{l=1}^m \frac{\partial \Omega^j(\gamma_x)}{\partial x^l} \zeta^l ds \right) 
$$

The second variation:

$$
\frac{1}{2} \mathcal{V}(\gamma_x)[z, z] 
$$

$$
= \int_0^t \left( \frac{1}{2} \sum_{k=1}^m \frac{\partial^2 \Psi_L(\gamma_x)}{\partial x^k \partial x^l} \zeta^k \zeta^l + \frac{1}{2} \sum_{ij=1}^m \frac{\partial^2 g_{ij}(\gamma_x)}{\partial x^k \partial x^l} (\gamma_x^i + \Omega^i(\gamma_x))(\gamma_x^j + \Omega^j(\gamma_x)) + \sum_{j=1}^m p_j \frac{\partial^2 \Omega^j(\gamma_x)}{\partial x^k \partial x^l} \right) ds + 
$$

$$
+ \int_0^t \left[ \sum_{ij=1}^m \frac{\partial g_{ij}(\gamma_x)}{\partial x^k} (\gamma_x^i + \Omega^i(\gamma_x)) \left( d\zeta^j + \sum_{l=1}^m \frac{\partial \Omega^j(\gamma_x)}{\partial x^l} \zeta^l ds \right) 
$$

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Using the stochastic equation for $z_1$, we get
\[
\sqrt{2} \int_0^t \sum_{ijkl=1}^m \frac{\partial g_{ij}(\gamma_x)}{\partial x^k} (\gamma_x'^i + \Omega^i(\gamma_x)) \sigma^{ij}(\gamma_x) z_1^k dw_l = \\
\int_0^t \sum_{ijkl=1}^m \frac{\partial g_{ij}(\gamma_x)}{\partial x^k} (\gamma_x'^i + \Omega^i(\gamma_x)) \left( dz_1^j + \sum_{l=1}^m \frac{\partial \Omega^j}{\partial x^l}(\gamma_x) z_1^l ds \right)
\]

\[
\frac{1}{2} \nu(\gamma)[z, z] = I_2(t; z) + \\
\int_0^t \sum_{ij=1}^m g_{ij}(\gamma_x)(z'^i + \sum_{l=1}^m \frac{\partial \Omega^i}{\partial x^l}(\gamma_x) z'^l)(z'^j + \sum_{k=1}^m \frac{\partial \Omega^j}{\partial x^k}(\gamma_x) z'^k) ds - \int_0^t \sum_{k=1}^m p_k \hat{\Omega}^k(\gamma_x) ds
\]

Using considerations of paragraph 7.8 of [2] p274-275, we conclude that
\[
Pr\{ -I_2(t) \geq r \} \leq e^{-cr},
\]
where $c > 1$. This implies that there exists a $\beta > 0$ such that
\[
E_x (\exp -(1 + \beta) I_2(t)) < +\infty.
\]

**Estimate of $I_3(t)$**

We have
\[
I_{31}(t) = O(\sqrt{\epsilon R_{3,\epsilon}}),
\]
\[
I_{32}(t) = O(\sqrt{\epsilon R_{2,\epsilon}^2} + (\sqrt{\epsilon} R_{2,\epsilon})^2),
\]
\[
I_{33}(t) = O(\sqrt{\epsilon} R_{1,\epsilon}^3),
\]
\[
I_{34}(t) = \int_0^t O(\sqrt{\epsilon} R_{1,\epsilon}^2) dw,
\]

where if $X_s, Y_s$, $s \in [0, T]$ are two processes, we say that $Y_s = O(X)$ for $s \in [0, t]$ if $Y_s \leq K X$, for some positive constant $K$, and where $X = \sup_{s \in [0, t]} |X_s|$. Using the same consideration as in [2] p 270-271, paragraph 7.8, we conclude that there exists a function $\rho(\alpha)$, defined for all positive $\alpha$ small enough, such that if the radius $\eta$ of the ball $E_3$ is smaller then $\rho(\alpha)$, then
\[
E_x \left( e^{(1+\alpha) I_3(t)} \right) \leq C.
\]

We conclude that there exists a constant $C > 0$ such that
\[
E_x \left\{ \chi_3(X_{\epsilon}) e^{-\frac{I_3(X_{\epsilon}(s))}{\epsilon}} \right\} \leq C \exp \left\{ -\frac{I_0(t)}{\epsilon} \right\}.
\]
References


